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INTERSECTION CLASSES AND MULTIPLE INTERSECTION PARAMETERS OF GRAPHS

Princeton University PH.D. 1984

University
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International 300 N. Zeeb Road, Ann Arbor, MI 48106

INTERSECTION CLASSES AND MULTIPLE INTERSECTION PARAMETERS OF GRAPHS

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A DISSERTATION
PRESENTED TO THE FACULTY
OF PRINCETON UNIVERSITY
IN CANDIDACY FOR THE DEGREE
OF DOCTOR OF PHILOSOPHY

RECOMMENDED FOR ACCEPTANCE
BY THE DEPARTMENT OF
MATHEMATICS

June, 1984

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ACKNOWLEDGEMENTS

This thesis appears at the end of one phase of my mathematics career and at the beginning of another. I would like to mark this transition by thanking some of the people who have made important contributions to my becoming a mathematician: Susan and Ira Scheinerman, Peggy Karlin, Debbie (Sands) Lubell, Kenneth Rosenthal, John Wells, Andrew Mendelsohn, David Kelly, Peter Miletta, Dr. Sylvan, Frank Stewart, Michael Rosen, Thomas Banchoff and Shimshon Amitsur.

In addition, I want to give special thanks to Doug West. He has been my advisor and my friend through difficult situations and over many miles. I want to thank Hale Trotter for his specific contributions to this thesis. Thanks also to Tom Trotter, Avi Wigderson, Jeff Kahn and Michael Saks for many interesting discussions on topics related to this thesis. I would also like to thank Andy Ross for his assistance with the art work.

Most importantly, I want to thank my wife, Amy. Her love and encouragement supported me throughout this project. She and our daughter, Rachel, are my greatest source of delight and happiness. I am fortunate beyond measure to have them and be theirs.

ABSTRACT

Let **S** denote a family of sets, t**S** the t-fold unions of sets in **S** and Ω(**S**) the class of intersection graphs of **S**. Put **S**#(G)=inf{t:G$\in\Omega$(t**S**)}. When **S** is the family of real intervals, Ω(**S**) is the class of interval graphs and **S**# is the interval number. Our principal results include:

- A readily applicable characterization of those classes of graphs of the form Ω(**S**).
- Testing G$\in\Omega$(**S**) can have arbitrary computational complexity.
- Asymptotic calculations on depth-r interval number of complete graphs.
- The devastating effect of a non-redundancy restriction on multiple interval representations.
- Analysis of boundedness of **S**# parameters. In particular, **S**# is bounded if and only if Ω(**S**) contains all bipartite graphs.
- Asymptotic calculation of maximum value of interval number for graphs with given genus.
- Calculation of the following upper bounds for the **S**# of planar graphs: For **S**={real intervals}, 3; for **S**={planar boxes}, 2; for **S**={planar line segments}, 2.

Many results are generalized to embrace uniform hypergraphs and simplicial complexes.

"Beauty is truth, truth beauty" - that is all
Ye know on earth, and all ye need to know.

John Keats, *Ode on a Grecian Urn*

The Mishnah states: If a fledgling bird is found within fifty cubits of a dovecote, it belongs to the owner of the dovecote. If it is found outside the limit of fifty cubits, it belongs to the person who finds it.

Rabbi Jeremiah asked: If one foot of the fledgling is within fifty cubits, and one foot is outside it, what is the law?

It was for this question that Rabbi Jeremiah was thrown out of the House of Study.

Baba Batra 23b

0. PRELIMINARIES

This chapter introduces the basic concepts and notation we employ in this dissertation.

0.1 Sets, Multisets

0.1.1 Notation. **Sets** are denoted by curly braces: $\{\}$. The words **collection** and **family** are synonymous with set and are used especially when the elements themselves are sets. When X is a set and k is a non-negative integer, $\binom{X}{k}$ denotes the family of all k-element subsets of X and 2^X denotes the set of all subsets of X. The cardinality of X is denoted $|X|$. The symbol $\subset$ indicates subset without ruling out equality.

0.1.2 Definition. A **multiset** is a pair $M=(X,f)$ where X is a set and f a function $f:X \rightarrow Z^+$. The function f counts the number of times a given element is in the multiset.

We use square brackets to denote a multiset: $M=[x_j(m_j):j\in J]$ is the multiset (X,f) where $X=\{x_j:j\in J\}$ and $f(x_j) = m_j$ for $j\in J$.

0.2 Graphs

0.2.1 Definition. A **graph** is an ordered pair of finite sets $G=(V,E)$ with $E\subset\binom{V}{2}$.

In other words, by a graph we mean a simple, finite, undirected graph without loops or multiple edges. For any graph G we use $V(G)$ and $E(G)$ to denote the vertex and edges sets of G. Often we use the letters n and m for $|V(G)|$ and $|E(G)|$ respectively.

If $e=\{x,y\}\in E$ we say that x is **adjacent** to y and we write $x\sim y$. We may write $e=xy$ for edges.

The set of all vertices adjacent to a vertex v is called the **neighborhood** of v and is denoted

$$\mathrm{adj}(v) = \{w: w\sim v\}.$$

0.2.2 Definition. For graphs G and H we say that G is a **subgraph** of H provided $V(G)\subset V(H)$ and $E(G)\subset E(H)$. If G satisfies the property that $E(G)=E(H)\cap\binom{V(G)}{2}$ we say that G is an **induced** subgraph of H and we write $G\leq H$ or $H\geq G$. In case $|V(G)|<|V(H)|$ we may write $G<H$ or $H>G$.

0.2.3 Definition. A class of graphs $\mathbf{P}$ is said to be **monotone** if it is closed under taking induced subgraphs, i.e.,

$G\in\mathbf{P}$ and $H\leq G$ imply that $H\in\mathbf{P}$.

0.2.4 Definition. If G and H are graphs, we say G is **isomorphic** to H provided there exists a bijection $f:V(G)\longrightarrow V(H)$ such that $xy\in E(G)$ if and only if $f(x)f(y)\in E(H)$.

Often we do not distinguish between isomorphic graphs. We will consider them to be the same graph. For example, when we write $G\leq H$ we mean that G is isomorphic to an induced subgraph of H.

0.2.5 Notation. Let G and H be graphs. We define $G\cup H$ to be the graph $(V(G)\cup V(H),E(G)\cup E(H))$ and $G\cap H$ to be the graph $(V(G)\cap V(H),E(G)\cap E(H))$. However, unless we indicate otherwise, we will assume that $V(G)=V(H)$ when forming graph unions and intersections. One notable exception is the following:

The **disjoint union** of G and H, denoted G+H, is the graph formed by assuming V(G) and V(H) are disjoint (e.g., by taking isomorphic copies

of G and H on disjoint sets of vertices) and then forming their union. We denote by nG the n-fold disjoint union of G with itself: G+G+...+G.

0.2.6 Definition. Let G and H be graphs. The **join** of G and H, denoted G∨H, is defined to be the graph formed from G+H by adding edges connecting every vertex in G to every vertex in H.

0.2.7 Remark. We use other graph theory terms that are not explicitly defined in this dissertation. Terms such as path, cycle, connected, component, tree, forest, degree, spanning, complete graph, independent set, etc. which are frequently used in graph theory literature have their usual meanings. Cliques are complete subgraphs and need not be maximal. Definitions can be found in either [35] or [5].

0.3 Hypergraphs and Simplicial Complexes

0.3.1 Definition. A **hypergraph** H is a pair of finite sets (V,E) with $E\subset 2^V-\{\emptyset\}$. If for some positive integer k we have $E\subset\binom{V}{k}$ then the hypergraph is said to be **k-uniform**. Graphs are exactly the 2-uniform hypergraphs.

0.3.2 Definition. A (finite) **simplicial complex** is a hypergraph K=(V,E) that satisfies the properties:

(1) if $\emptyset\neq e'\subset e\in E$ then $e'\in E$, and

(2) $v\in V$ implies $\{v\}\in E$.

Normally one does not distinguish between v and {v} for vertices of a simplicial complex.

0.3.3 Definition. If K is a simplicial complex, the **dimension** of K is defined by

$$\dim(K) = \max\{|e|:e\in E(K)\} - 1.$$

The edges of cardinality k+1 are called **k-simplices**. For each integer

$k \geq 0$, define the **k-skeleton** of K to be the simplicial complex $sk_k K$ given by $V(sk_k K)=V(K)$ and $E(sk_k K)=\{e \in E: |e| \leq k+1\}$.

There is an obvious bijective correspondence between graphs and 1-dimensional simplicial complexes. In particular the 1-skeleton of a simplicial complex is considered a graph.

0.3.4 Definition. If K is a simplicial complex we define the **barycentric subdivision** of K, denoted βK, to be the simplicial complex defined by:

(1) $V(\beta K) = E(K)$, and

(2) $E(\beta K) = \{ \{e_1, e_2, \ldots, e_j\}: e_1 \subset e_2 \subset \ldots \subset e_j\}$

When G is a graph βG is the graph formed by subdividing the edges of G in the usual graph theoretic sense. See figure 0.1.

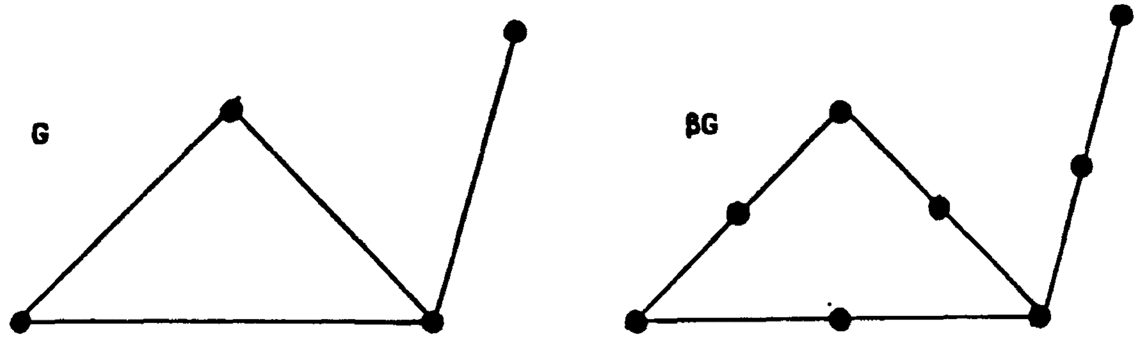


Figure 0.1

0.3.5 Definition. Let k be a positive integer. A hypergraph is said to be **k-partite** if and only if there exists a partition of the vertex set into (at most) k (disjoint) sets, called parts, such that no edge contains more than one vertex from any part. In case $k=2$, the

hypergraph is called bipartite.

0.4 Posets

0.4.1 Definition. A [finite] **partially ordered set** or **poset** is a pair $P=(X,\leq)$ where X is a [finite] set and $\leq$ is a relation on X satisfying for all $x,y,z \in X$:

(1) $x \leq x$,

(2) $x \leq y$ and $y \leq x$ imply $x=y$, and

(3) $x \leq y$ and $y \leq z$ imply $x \leq z$.

If for all $x,y \in X$ we have $x \leq y$ or $y \leq x$ then P is called a **total order**.

0.4.2 Definition. If $P_1=(X,\leq_1)$ and $P_2=(X,\leq_2)$ are posets then $P_1 \cap P_2$ is the poset $(X,\leq)$ where $x \leq y$ if and only if $x \leq_1 y$ and $x \leq_2 y$.

0.4.3 Definition. If $P=(X,\leq)$ is a poset, the **dimension** of P is the least positive integer t such that $P = P_1 \cap P_2 \cap \ldots \cap P_t$ with each P_i a total order for $1 \leq i \leq t$. It is easy to see that the dimension of any finite poset is finite.

0.4.4 Definition. If $P=(X,\leq)$ is a partial order with $x,y \in X$, the **interval** $[x,y]$ is the set $\{z \in X : x \leq z \text{ and } z \leq y\}$.

0.4.5 Definition. Let $(X,\leq)$ be a poset. We call a subset Y of X an **ideal** of the poset if $y \in Y$ and $x \leq y$ imply $x \in Y$.

Example: The class of all graphs **G** together with the relation of induced subgraph form a poset. The ideals of $(\mathbf{G},\leq)$ are exactly the monotone classes of graphs.

0.5 Ramsey Theory

We make use of the following theorems from Ramsey Theory. Proofs can be found in [28].

0.5.1 Theorem [Ramsey]. Let k_1 and k_2 be positive integers. Then there exists a number N such that for every coloring of the edges of K_N with two colors, blue and red, there exists a subgraph of K_N that is either a blue k_1-clique or a red k_2-clique.∎

0.5.2 Definition. We denote the least N satisfying the conditions of the above theorem by $r(k_1,k_2)$. The numbers $r(k_1,k_2)$ are called **Ramsey numbers.**

0.5.3 Theorem [Edge Induced Ramsey]. Let G be a graph and t be a positive integer. Then there exists a graph H such that for every t coloring of the edges of H there exists an induced copy of G in H in which all the edges have the same color.∎

In this theorem, if G is bipartite, we may require H to be bipartite as well. This follows by setting k=2 in the following result:

0.5.4 Theorem. Let k and t be positive integers and let G be a k-partite k-uniform hypergraph. Then there exists a k-partite k-uniform hypergraph H such that for every t coloring of the edges of H there exists an induced copy of G in H such that all the edges of G have the same color.∎

0.6 Computational Complexity

In this thesis we discuss the computational complexity of determining membership in various intersection classes. The concepts of

polynomial transformation ($\propto$), NP-completeness and other related concepts are defined and discussed in [21].

0.7 Symbols

$\mathbf{Z}$	Integers
$\mathbf{Z}^+$	Positive Integers
$\mathbf{R}$	Reals
$\mathbf{R}^d$	Euclidean d-space
$\mathbf{RP}^d$	d-dimensional projective space
$\mathbf{G}$	Class of all graphs
K_n	Complete graph on n vertices
$K_{n,m}$	Complete bipartite graph
$K(n_1,\ldots,n_t)$	Complete multipartite graph
$d_G(v)$	Degree of a vertex
$\Delta(G)$	Maximum degree
$\overline{G}$	Graph complement
$L(G)$	Line graph
$\chi(G)$	Chromatic number
$\alpha(G)$	Independence number
$\omega(G)$	Size of largest clique
$\gamma(G)$	Genus
$\kappa(G)$	Connectivity
$\mathbf{S}$	A family of sets
$\mathbf{P}$	A class of graphs
$\lfloor x \rfloor$	Greatest integer $\le$ x
$\lceil x \rceil$	Least integer $\ge$ x
$x<<y$	x much less than y
PL	Piece-wise linear
χ	Euler characteristic
o	little "oh": $\lim_{x\to\infty}[o(x)/x]=0$
2^X	Power set (§0.1.1)

$\binom{X}{k}$	(§0.1.1)
$v \sim w$	Vertex adjacency (§0.2.1)
adj(v)	Neighborhood (§0.2.1)
$G \leq H$, $G < H$	Induced, proper induced subgraph (§0.2.2)
$G \cup H$, $G \cap H$	Graph union, intersection (§0.2.5)
$G+H$, nG	Disjoint union (§0.2.5)
$G \vee H$	Join (§0.2.6)
dim(K)	Simplicial complex dimension (§0.3.3)
$sk_k K$	k-skeleton (§0.3.3)
βG, βK	Subdivision (§0.3.4)
$P_1 \cap P_2$	Poset intersection (§0.4.3)
$[x,y]$	Interval (§0.4.4)
$r(k_1,k_2)$	Ramsey number (§0.5.2)
$\Pi_1 \propto \Pi_2$	Polynomial Transformation (§0.6)
NP	Nondeterministic Polynomial (§0.6)
$\Omega(S)$	Intersection class (§1.1.2)
$\Omega([\text{multiset}])$	Intersection graph (§1.1.2)
$\mathbf{B}^d$	Boxes in $\mathbf{R}^d$ (§1.1.16)
b(G)	Boxicity (§1.1.17)
PB(G)	Poset boxicity (§1.1.20)
$\mathbf{K}^d$	Compact convex sets in $\mathbf{R}^d$ (§1.1.21)
$v \equiv w$	Vertex equivalence (§1.2.2)
ρG	Reduced graph (§1.2.3)
$\rho^{-1}(G)$	(§1.2.4)
ιG	Isolated vertex removal (§1.2.21)
$\Omega_1(S)$	Injective intersection class (§1.3.2)
N(S)	Nerve (§1.4.1)
$\mathbf{N}(S)$	Family of nerves (§1.4.4)
$Q(k_1,\ldots,k_t)$	Q-graph (§1.5.4)
$\Gamma(n)$	(§1.5.5)
πG	Pruned subgraph (§1.5.6)
Π_P	(§1.5.12)

Π_A	(§1.5.12)
tS	t-fold unions (§2.1.1)
$S\#$	(§2.1.2)
$i(G)$	Interval number (§2.1.5)
$i^+(G)$	Displayed interval number (§2.1.14)
G^+	(§2.1.15)
$i_r(G)$	Depth-r interval number (§2.1.17)
$i_r^+(G)$	Depth-r displayed interval number (§2.1.17)
$T(G)$	Arboricity (§2.2.5)
Uf	(§2.2.14)
$I(G)$	Intersection number (§2.2.14)
$G \times H$, G^k	Graph product, exponent (§2.2.22)
$\tau(f)$	(§3.2.5)
$L^*(t)$	(§3.2.6)
$NRI(G)$	Non-redundant interval number (§3.3.2)
$mult(f)$	Multiplicity (§4.2.1)
$P\#$	(§4.2.3, §4.5.2)
LS^d	Line segments in R^d (§4.3.3)
$G \vee_2 H$	Bipartite join (§4.3.8)
$P_\cup$, $P_\cap$	(§4.4.1)
$\bar{P}$	(§4.4.5)
$F(P)$	(§4.4.5)
$P_h\#$	(§4.4.10)
Ω^*, Ω'	(§4.5.5)
N^*, N'	(§4.5.7)
u	(§4.5.8)
$\mathcal{S}$, $\mathcal{M}$, $\mathcal{U}$	(§4.5.9)
σ	(§4.5.9)
μ	(§4.5.9)
oG	Outer induced subgraph (§5.1.2)
G^*	Planar dual (§5.1.4)
wG	Weak dual (§5.1.4)

bG	Block graph (§5.1.7)
a(g)	(§5.1.14)
$\rho(v,w)$, $\rho(v)$, ρ^*	Redundancy in representation (§5.2.2)
$\beta(v)$, β^*	Broken ends in representation (§5.2.1)
ξ	Reusable endpoint assignment (§5.3.7)
imax(g)	$\sup\{i(G)\colon\ \gamma(G)=g\}$ (§5.4.3)

1. INTERSECTION REPRESENTATIONS

In this chapter we introduce the central concept of this dissertation: intersection representations of graphs. Many classes of graphs are defined as intersection graphs of sets of various sorts such as arbitrary sets, intervals in $\mathbf{R}$, circular arcs, etc. This chapter's main result describes what all these classes have in common—we characterize intersection classes.

We begin in section 1.1 by defining intersection representations and intersection classes. We then discuss several examples of intersection classes, the most important of which being the class of interval graphs. In section 1.2 we characterize intersection classes, i.e., we give necessary and sufficient conditions that a given class of graphs must satisfy in order to be an intersection class. In section 1.3 we discuss and characterize injective intersection classes. Section 1.4 discusses nerves, the generalization of intersection graphs to simplicial complexes. We give a characterization of nerve classes. We conclude with section 1.5 in which we discuss, in a general setting, computational complexity problems associated with intersection classes.

1.1 Intersection Representations

1.1.1 Definition. Let $\mathbf{S}$ be a family of sets and let G be a graph. We say that G has an **intersection representation** by sets in $\mathbf{S}$ (or an $\mathbf{S}$-**representation**) if there exists a function $f:V(G)\rightarrow\mathbf{S}$ such that for all $v,w\in V(G)$ with $v\neq w$ we have $v\sim w$ if and only if $f(v)\cap f(w)\neq\emptyset$. Note that the function f need not be one-to-one.

1.1.2 Definition. We denote the class of graphs that have an S-representation by $\Omega(\mathbf{S})$. Such a family is called an **intersection class** of graphs.

Let M be a finite multiset of sets of $\mathbf{S}$, $M=[S_i(m_i):i\in J]$ with each $S_i\in\mathbf{S}$. We define $\Omega(M)$ to be the graph with Σm_i vertices corresponding to the elements (sets) of M such that two vertices are adjacent if and only if the corresponding sets have nonempty intersection. If $G\in\Omega(\mathbf{S})$ and $f:V\rightarrow\mathbf{S}$ is an S-representation of G then we have $G=\Omega[f(v):v\in V(G)]$. The use of the symbol Ω in these two ways appears somewhat ambiguous, however, $\Omega(M)$ for a multiset of sets M is a graph, while $\Omega(\mathbf{S})$ for a set of sets $\mathbf{S}$ is a class of graphs.

Varying the choice of the family $\mathbf{S}$ results in various classes $\Omega(\mathbf{S})$. The following are examples of various intersection classes that have been studied in the literature.

1.1.3 Example: Arbitrary Sets. We wish to determine which graphs may be realized as intersection graphs of arbitrary sets. Clearly this will be the largest possible intersection class of graphs. In essence we want $\mathbf{S}$ to be the "set of all sets". Fortunately, it is enough to assume that $\mathbf{S}$ contains all subsets of the integers:

1.1.4 Theorem [51]. $\Omega(2^{\mathbf{Z}})=\mathbf{G}$, the class of all graphs.█

In other words, every graph has an intersection representation by arbitrary sets (of integers).

1.1.5 Example: Interval Graphs. We wish to determine the class of all intersection graphs of (closed) real intervals. In other words, $\mathbf{S}=\{[a,b]:a,b\in\mathbf{R},\ a<b\}$, and the class of interval graphs is $\Omega(\mathbf{S})$. The interval graphs are central in our discussion; virtually all the

questions we address concern generalizations of interval graphs.

The class of interval graphs has been extensively studied. There are several theorems giving good descriptions of this class as well as a linear time recognition algorithm. These are mentioned below.

It is not important whether the intervals are closed or open. Likewise, we do not have to consider the intervals as subsets of **R**—any total order will do. We now review the principal characterizations of interval graphs.

1.1.6 Definition. A graph G is called a **comparability graph** if it has a transitive orientation, i.e., if xy and yz are edges with $x \to y$ and $y \to z$ then $x \sim z$ and $x \to z$.

1.1.7 Definition. A graph is said to be **chordal** (or **triangulated** or **rigid circuit**) if it has no chordless cycles of length greater than 3.

Our first characterization is due to Gilmore and Hoffman:

1.1.8 Theorem [24]. A graph G is an interval graph if and only if G is a chordal graph and $\bar{G}$ is a comparability graph.∎

Gilmore and Hoffman also have the following characterization:

1.1.9 Theorem [24]. A graph G is an interval graph if and only if the maximal cliques of G can be linearly ordered such that the maximal cliques containing each vertex of G occur consecutively.∎

1.1.10 Definition. Three distinct vertices of a graph G are called an **asteroidal triple** if there exist paths joining each pair that contain no vertex in the neighborhood of the third.

The following result is due to Lekkerkerker and Boland:

1.1.11 Theorem [47]. A graph G is an interval graph if and only if G is chordal and contains no asteroidal triple.∎

Lekkerkerker and Boland also give a complete list of minimal forbidden subgraphs.

1.1.12 Theorem [47]. A graph G is not an interval graph if and only if it contains one of the graphs shown in figure 1.1 as an induced subgraph.

We conclude with the following result of Booth and Leuker concerning the computational complexity of recognizing interval graphs.

1.1.13 Theorem [6,7]. Interval graphs can be recognized in linear time, i.e., there exists an algorithm which can determine if a given graph is an interval graph in at most $c(|V(G)|+|E(G)|)$ steps for some constant c.∎

The remaining examples of intersection classes may be appropriately viewed as generalizations of interval graphs in which "intervals" are generalized in some fashion.

1.1.14 Example: Circular Arc Graphs. If, instead of considering intervals on a line, we consider "intervals" on a circle, we are led to the concept of circular arc graphs. By an **arc** of a circle we mean, of course, a subset of a circle homeomorphic to a real interval. If S is the set of all arcs of a particular circle, then $\Omega(S)$ is the class of circular arc graphs. By deleting a point from the circle we see that every interval graph is also a circular arc graph. The converse, however, is clearly false; for example, C_4 is a circular arc graph which

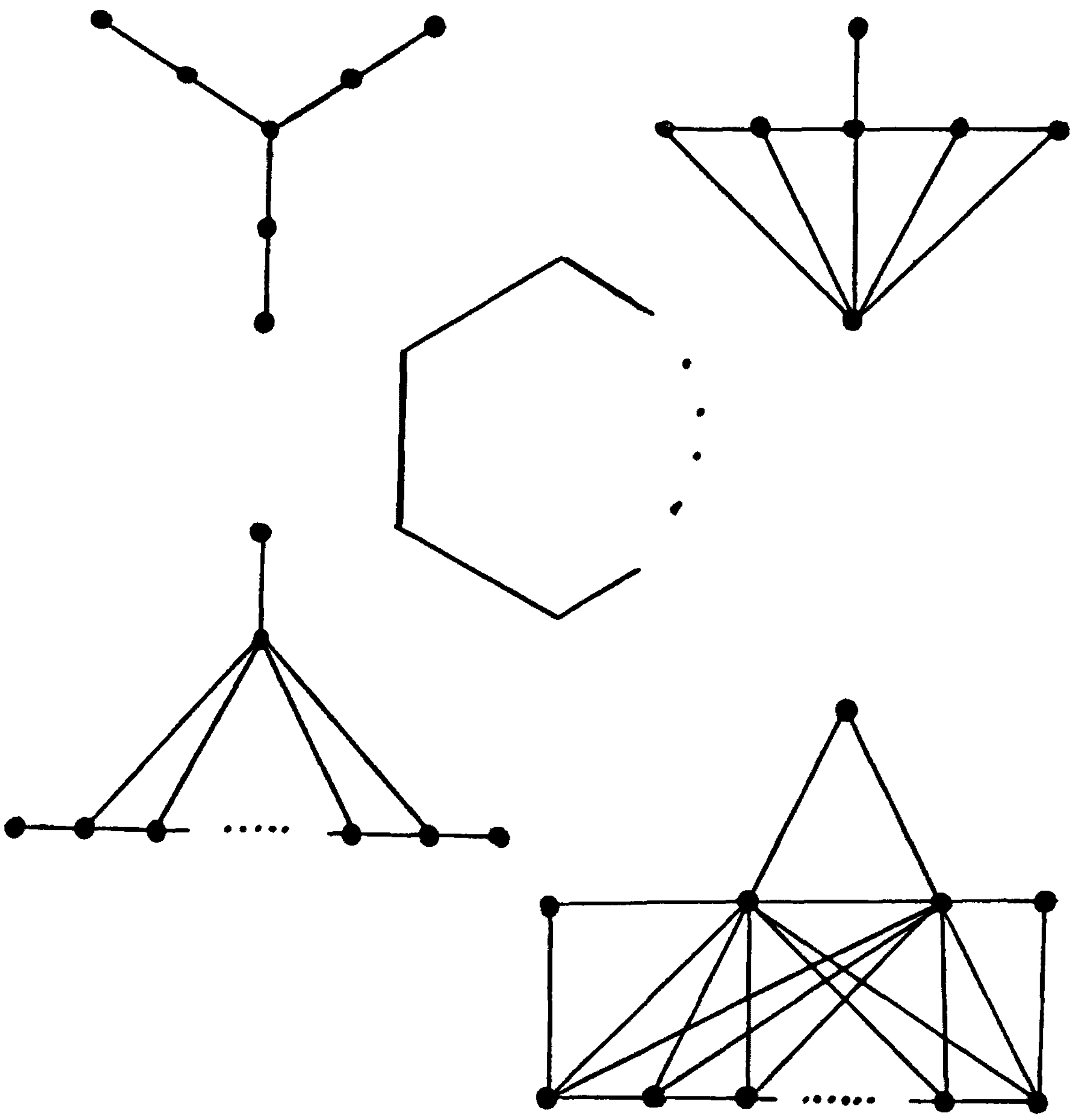


Figure 1.1

is not an interval graph.

Tucker has characterized circular arc graphs:

1.1.15 Theorem [70]. A graph G is a circular arc graph if and only if its vertices can be (cyclically) indexed $v_1, v_2, \ldots, v_n$ so that for all i,j we have $v_i \sim v_j$ implies either $v_{i+1}, \ldots, v_j$ are all in $\mathrm{adj}(v_i)$ or $v_{j+1}, \ldots, v_i$ are all in $\mathrm{adj}(v_j)$ with addition taken modulo n.∎

In addition, Tucker [74,75] has a polynomial time algorithm for the recognition of circular arc graphs.

1.1.16 Example: Boxicity. One way to generalize intervals is to consider sets in higher dimensional spaces which resemble intervals. For example, we may consider boxes in $\mathbf{R}^d$ (with sides parallel to the coordinate axes). A **box** in $\mathbf{R}^d$ is a set $\{(x_1, \ldots, x_d) : a_i \le x_i \le b_i,$ for $1 \le i \le d\}$ for fixed a_i and b_i. In other words, a box is the cartesian product of intervals: $[a_1,b_1] \times \ldots \times [a_d,b_d]$. Let $\mathbf{B}^d$ denote the set of all boxes in $\mathbf{R}^d$.

It can be shown easily that every graph belongs to $\Omega(\mathbf{B}^d)$ for d sufficiently large and also that no $\Omega(\mathbf{B}^d)$ contains all graphs. This leads to the definition of a graph parameter:

1.1.17 Definition. Let G be a graph. The **boxicity** of G, denoted b(G), is the least positive integer d for which $G \in \Omega(\mathbf{B}^d)$.

Boxicity has been investigated by several authors, especially Roberts. Cozzens' thesis [10] gives extensive discussion and investigation into boxicity. One result on boxicity which we need is the following:

1.1.18 Theorem [59]. The boxicity of the complete multipartite graph $K(n_1, \ldots, n_k)$ equals the number of the n_i which are greater than 1.∎

Note that some authors prefer to set the boxicity of the complete

graphs K_n to be 0. Since K_n=K(1,...,1) this is consistent with the above theorem. However, we prefer our definition with $b(K_n)=1$, especially in light of the following.

1.1.19 Proposition [59]. Let G=(V,E) be a graph and let t be the smallest integer for which there exist interval graphs $(V,E_1),\ldots,(V,E_t)$ such that $G=(V,E_1\cap\ldots\cap E_t)$. Then b(G)=t.∎

This is easily verified by projection onto coordinate axes.

One nice feature of Proposition 1.1.19 is that it provides an alternative definition for b(G) which makes no mention of coordinates or $\mathbf{R}^d$, however, the geometric flavor is lost. One idea to reconcile the geometric and the coordinate free aspects of the two alternatives is to note that the boxes in $\mathbf{R}^d$ are exactly intervals in $\mathbf{R}^d$ considered as a poset with $x\le y$ provided $x=(x_1,\ldots,x_d)$, $y=(y_1,\ldots,y_d)$ and $x_i\le y_i$ for i=1,...,d. Thus one could give a definition in which G must be represented by intervals in a minimal product of total orders. Such a definition would be equivalent to 1.1.17. Generalizing slightly further, one is lead to the following definition:

1.1.20 Definition. Let G be a graph. The **poset boxicity** of G, denoted PB(G), is the smallest positive integer t for which there exists a poset P with dim(P)=t, such that G has an intersection representation by intervals in P.

It is clear that PB(G)≤b(G) since G certainly has an interval representation in the poset $\mathbf{R}^{b(G)}$ which has dimension b(G). It would be very nice if PB(G)≥b(G), but this is false. We show that PB(K(2,2,2))<b(K(2,2,2)). By Theorem 1.1.18, b(K(2,2,2))=3. Now consider the poset depicted in figure 1.2. One checks that it has dimension 2 as it is the intersection of the total orders:

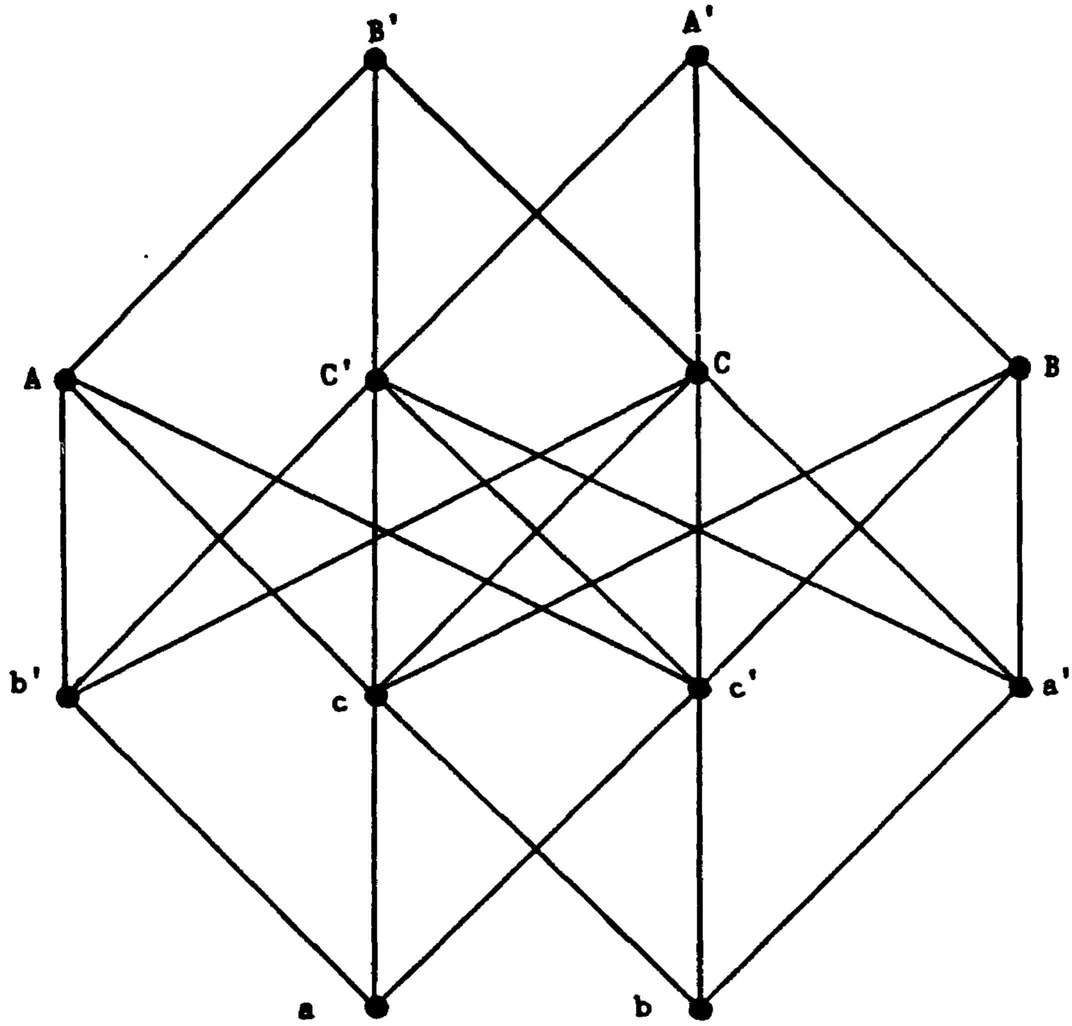


Figure 1.2

$$a<b<b'<c<c'<A<a'<C'<C<B'<B<A$$

and

$$b<a'<a<c'<c<B<b'<C<C'<A'<A<B'.$$

We see that the intersection graph of the intervals [a,A], [a',A'], [b,B], [b',B'], [c,C] and [c',C'] is K(2,2,2). Since K(2,2,2) is not an interval graph we must have $2=PB(K(2,2,2))<b(K(2,2,2))=3$.

1.1.21 Example: Convex Sets. Instead of generalizing intervals in higher dimesions as boxes, we consider convex sets. Let $\mathbf{K}^d$ denote the set of all compact convex sets in $\mathbf{R}^d$. Clearly $\mathbf{K}^1$=$\mathbf{B}^1$. The classes $\Omega(\mathbf{K}^d)$ are discussed in Wegner's thesis [77]. The principal results on these classes can be summarized as follows:

1.1.22 Theorem [77]. (a) $\Omega(\mathbf{K}^1)$ = the class of all interval graphs.

(b) $\Omega(\mathbf{K}^2)$ does not contain all graphs, but it does contain all planar graphs.

(c) $\Omega(\mathbf{K}^d)$ for $d \geq 3$ contains all graphs.∎

The "standard" example to show that $\Omega(\mathbf{K}^2)$ does not contain all graphs is βK_5 (the subdivision of K_5, see Definition 0.3.4). These results are more fully discussed in chapter 4. To date there is no characterization of $\Omega(\mathbf{K}^2)$.

1.1.23 Example: Curves, Line Segments. In the spirit of the previous examples, we can generalize intervals to higher dimensions as either curves or straight line segments. In dimension two it is known that not all graphs can be represented by curves: βK_5 is the usual example. This example first appeared in [63] but was not cited by later authors. Undoubtedly this was because it appeared in a paper entitled "Topology of thin film RC circuits". [12] showed that the intersection class formed by planar curves strictly contains the class formed by planar line segments. In dimension 3 and higher it is easy to see that the intersection class of curves contains all graphs. However, the intersection class of line segments in any real vector space (even with infinite dimension) does not contain all graphs. This is proved in chapter 4.

1.1.24 Example: Subtrees. Thus far we have viewed **R** as a topological space, as a poset and as a vector space in generalizing interval graphs. In the next few examples, we view **R** as a path graph. The simplest graph theoretic generalization of a path is a tree in which intervals become subtrees. There happens to be a simple characterization of the intersection graphs of subtrees of a tree.

1.1.25 Theorem [22]. The class of intersection graphs of subtrees of trees is exactly the class of chordal graphs.▮

1.1.26 Example: Paths in Trees—IPT and OPT. Instead of considering subtrees of a tree we now consider paths in trees. We may view paths either as a sequence of vertices or as a sequence of edges. In the first case, the intersection graphs we attain are called IPT graphs (Intersection of Paths in Trees) and in the latter case, OPT graphs (Overlap of Paths in Trees). The difference is that in IPT graphs paths which share only a single point correspond to adjacent vertices, while in OPT graphs those vertices would not be adjacent. See figure 1.3. IPT graphs have been investigated by [58] and [23], and OPT graphs by [26].

If, instead of considering paths in trees, we allow the paths to be in arbitrary graphs we arrive, by analogy, with the classes IPG and OPG. These classes turn out to be trivial.

1.1.27 Theorem. Every graph is an IPG and an OPG graph.

Proof. The following construction applies in both cases. Let G be an arbitrary graph. Let $V(G)=\{v_1,\ldots,v_n\}$ and $E(G)=\{e_1,\ldots,e_m\}$.

Now construct a graph H as follows. (H will have $4m+n$ vertices.) First, take $2m$ vertices $x_1,\ldots,x_m$ and $y_1,\ldots,y_m$ and join x_i to y_i by an edge for $i=1,\ldots,m$. Next, for each vertex $v\in V(G)$ add $\deg_G(v)+1$ new

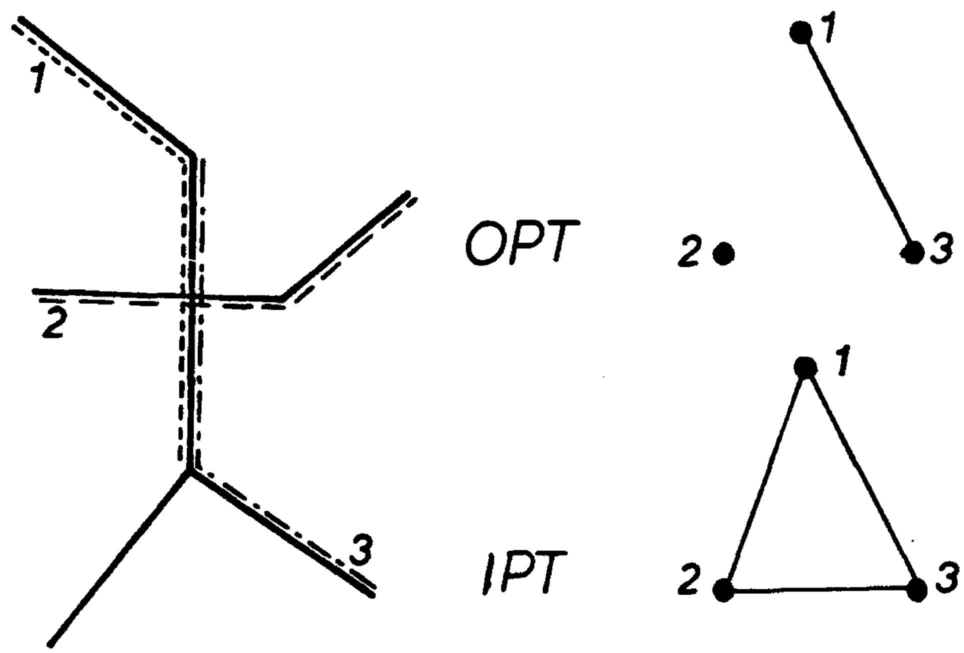


Figure 1.3

vertices $W_v=\{w_0,\ldots,w_d\}$. If vertex v is incident with edges $e_{i_1},\ldots,e_{i_d}$ in G with $i_1<\ldots<i_d$, then in H join x_{i_j} to w_{j-1} and join y_{i_j} to w_j by edges. This completes the construction of H. See figure 1.4.

Next, assign to each $v\epsilon V(G)$ a path P_v in H with P_v determined by the vertices $w_0 x_{i_1} y_{i_1} w_1 x_{i_2} y_{i_2} \ldots w_d$. Notice that two paths P_u and P_v intersect exactly along an edge of the form $x_j y_j$ if and only if $e_j=uv$.∎

1.1.28 Examples. There are still many more examples of intersection families which we could discuss. For example, circle graphs are the intersection graphs of chords of a circle, function graphs are the intersection graphs of the (cartesian) graphs of continuous function $f:[0,1]\rightarrow\mathbf{R}$, and permutation graphs are function graphs in which the functions f must be affine. See [25] for more information.

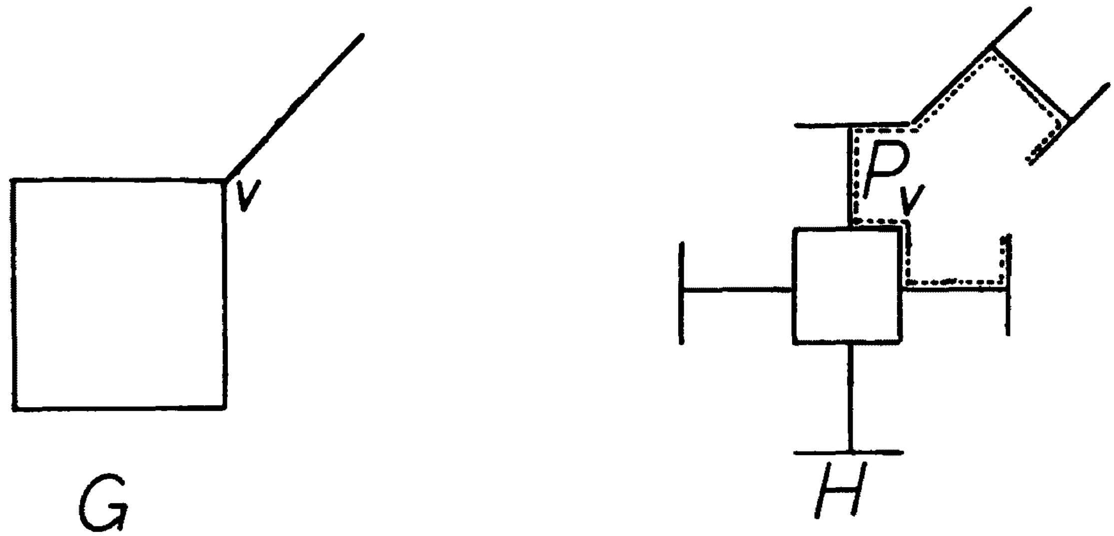


Figure 1.4

The central examples for this dissertation, however, are the multiple interval graphs which are defined and discussed in chapter 2.

1.2 Characterizing Intersection Classes

In the previous section we discussed numerous examples of intersection classes of graphs. In this section we discuss what properties all these classes share. Put more precisely, we answer the question: given a family **P** of graphs, when does there exist a family of sets **S** such that **P**=Ω(**S**)?

The answer to this question depends, in part, on whether or not the empty set is allowed to be a member of **S**. The reason for this minor anomaly is simple: normally two vertices assigned to the same set are adjacent unless the set in question is empty, in which case those vertices are isolated. Thus we first characterize the families Ω(**S**) for which $\emptyset$ is not in **S** and then we analyze the effect of adding the empty set to **S**.

We begin with a few lemmas giving necessary conditions for $P=\Omega(S)$.

1.2.1 Lemma. If $P=\Omega(S)$, then P is a monotone class of graphs (closed under taking induced subgraphs).

Proof. Suppose $H \le G \in P=\Omega(S)$. Let $f:V \to S$ be an S-representation for G. Clearly $f|V(H)$ is an S-representation for H and thus $H \in P$.∎

1.2.2 Definition. Two vertices u,v of a graph G are called **equivalent** provided that either u=v or else u~v and for all vertices $w \in V(G)$ with $w \ne u$, $w \ne v$ we have w~u if and only if w~v. We denote vertices u,v as equivalent by $u \equiv v$. In symbols,

$$u \equiv v \iff adj(u) \cup \{u\} = adj(v) \cup \{v\}.$$

Such vertices (with $u \ne v$) are also known as **true twins**.

It is trivial to verify that $\equiv$ is an equivalence relation.

1.2.3 Definition. Let G be a graph. Define ρG to be the **reduced** graph of G whose vertices are equivalence classes of vertices of G modulo $\equiv$ which are adjacent if and only if some element of one class is adjacent to some element of the other.

By the way in which equivalent vertices were defined it is unimportant which vertices we choose from each of a pair of classes to check if the classes (vertices of ρG) are adjacent.

Alternatively, we could define ρG as the induced subgraph of G formed by choosing one vertex per equivalence class.

Also note that $\rho\rho G = \rho G$.

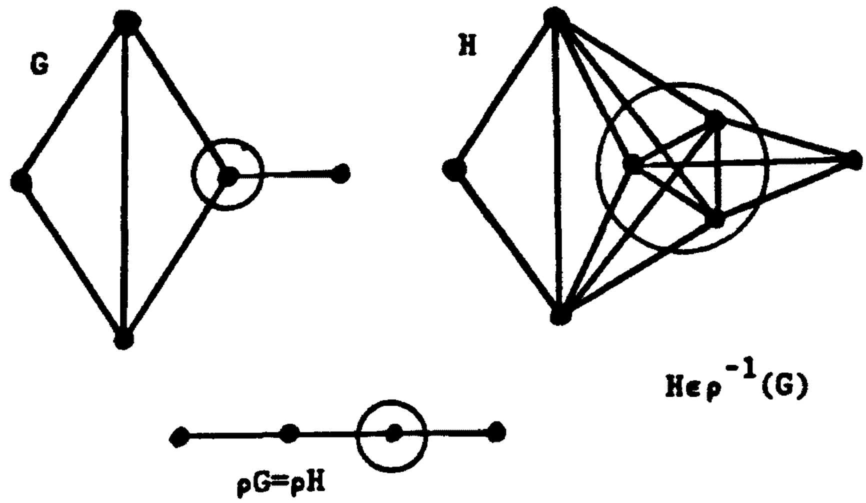


Figure 1.5

1.2.4 Definition. Let G be a graph. Define $\rho^{-1}(G)$ to be the class of graphs: $\{H: H\geq G, \rho H=\rho G\}$. If $H\epsilon\rho^{-1}(G)$ we say that H arises from G by **vertex expansion**. (See figure 1.5.)

1.2.5 Lemma. Let **S** be a family of nonempty sets and let $\mathbf{P}=\Omega(\mathbf{S})$. If $G\epsilon\mathbf{P}$ then $\rho^{-1}(G)\subset\mathbf{P}$.

Proof. Let $H\epsilon\rho^{-1}(G)$. Since $\rho H=\rho G\leq G$, and since by 1.2.1 **P** is monotone, $\rho H\epsilon\mathbf{P}$. Therefore, let $\rho H=\Omega[S_1,\ldots,S_n]$ with $S_i\epsilon\mathbf{S}$. Now for each $v_i\epsilon V(\rho H)$, $i=1,\ldots,n$ we have $m_i\geq 1$ vertices in H equivalent to v_i: $\{v_i^k: 1\leq k\leq m_i\}$. Now $v_i^p\sim v_j^q$ in H if and only if $v_i\sim v_j$ in ρH. Since no $S_i=\emptyset$, $H=\Omega[S_1(m_1),\ldots,S_n(m_n)]$, and therefore $H\epsilon\mathbf{P}$. It follows that $\rho^{-1}(G)\subset\mathbf{P}$.∎

1.2.6 Remark. Since we do not distinguish between isomorphic graphs, there are finitely many graphs with n vertices for any positive integer n. Hence the class of all graphs is countable.

1.2.7 Lemma. Let S be a family of sets and $P=\Omega(S)$. Then there exists a countable subset $S'\subset S$ such that $P=\Omega(S')$.

Proof. By 1.2.6 we may assume that P is countable, $P=\{G_1,G_2,G_3,\ldots\}$. Let $f_i:V(G_i)\to S$ be an S-representation of G_i. Let $S_i=f_i(V(G_i))=\{f_i(v_1),\ldots,f_i(v_n)\}$. Clearly S_i is a finite subset of S. Let $S'=S_1\cup S_2\cup S_3\cup\ldots$. Hence $S'\subset S$ and S' is countable. Since $S'\subset S$ clearly $\Omega(S')\subset\Omega(S)$. However, if $G\in\Omega(S)$, then $G=G_i$ for some i, hence $G\in\Omega(S_i)\subset\Omega(S')$. Therefore $\Omega(S)\subset\Omega(S')$.∎

1.2.8 Definition. Let P be a class of graphs. We say that P has a **composition series** if there exist graphs $G_1\le G_2\le G_3\le\ldots$ with $G_i\in P$ such that if $G\in P$ then $G\le G_k$ for some k.

Note that the G_i need not be distinct and that a composition series may be finite.

1.2.9 Lemma. If $P=\Omega(S)$, then P has a composition series.

Proof. By 1.2.7 we may assume that S is countable and that $S=\{S_1,S_2,S_3,\ldots\}$. Put $G_k=\Omega[S_1(k),S_2(k),\ldots,S_k(k)]$. Clearly $G_k\in P$ and $G_k\le G_{k+1}$ for $k=1,2,3,\ldots$. Let $G\in P$. Then there is a function $f:V(G)\to S$ which is an S-representation of G. Let

$k_1 = |V(G)|$ and

$k_2 = \max\{i:f(v)=S_i,\ v\in V(G)\}$.

Put $k=\max\{k_1,k_2\}$. Clearly $G\le G_k$ and therefore the G_i form the required composition series.∎

The results 1.2.1, 1.2.5 and 1.2.9 are each necessary conditions for a family P to be an intersection class of a family of nonempty sets. Our main result shows that these condition together are sufficient:

1.2.10 Theorem. Let $\mathbf{P}$ be a nonempty family of graphs. There exists a family of nonempty sets satisfying $\mathbf{P}=\Omega(\mathbf{S})$ if and only if the following conditions hold:

(i) $\mathbf{P}$ is monotone,

(ii) $G\in\mathbf{P}$ implies $\rho^{-1}(G)\subset\mathbf{P}$, and

(iii) $\mathbf{P}$ has a composition series.

Proof. By 1.2.1, 1.2.5 and 1.2.9 we see that if such an $\mathbf{S}$ exists then (i), (ii) and (iii) must hold. Conversely, suppose the three conditions hold. Let $G_1\le G_2\le G_3\le\ldots$ be a composition series for $\mathbf{P}$. Since $\mathbf{P}$ is monotone we may include in the series graphs "between" G_i and G_{i+1}. We can "refine" the series and we therefore may assume that $|V(G_i)|=i$.

Since $G_i\le G_{i+1}$ we may assume that $V(G_i)=\{v_1,v_2,\ldots,v_i\}$.

We now define sets S_i which consist of ordered pairs of positive integers as follows:

$S_i=\{(i,j):j>0,\ j\in\mathbf{Z}\}\cup\{(j,i):0<j<i,\ j\in\mathbf{Z},\ v_j\sim v_i\}$.

Note that in the second part of the definition of S_i we need only check vertices in G_i for adjacency to v_i. Let $\mathbf{S}=\{S_1,S_2,S_3,\ldots\}$. We claim that $\mathbf{P}=\Omega(\mathbf{S})$.

We first show that $G_k=\Omega[S_1,\ldots,S_k]$. Suppose $j<i\le k$ and $v_i\sim v_j$. It follows that $(j,i)\in S_i$ since $v_j\sim v_i$. Thus $S_i\cap S_j\ne\emptyset$. Conversely, suppose $j<i\le k$ and $(p,q)\in S_i\cap S_j$. Since $(p,q)\in S_i$ either $p=i$ or $q=i$. Since $(p,q)\in S_j$ either $p=j$ or $q=j$. These imply that either $(p,q)=(i,j)$ or $(p,q)=(j,i)$. Since $j<i$ we cannot have $(i,j)\in S_j$. Thus $S_i\cap S_j=\{(j,i)\}$ which implies $(j,i)\in S_i$ and so $v_j\sim v_i$. Thus $f:V(G_k)\to\mathbf{S}$ by $f(v_i)=S_i$ is an $\mathbf{S}$-representation of G_k and $G_k\in\Omega(\mathbf{S})$.

Now if $G\in\mathbf{P}$ we know by (iii) that $G\le G_k$ for some k, and since $\Omega(\mathbf{S})$ is monotone we have $G\in\Omega(\mathbf{S})$. Thus $\mathbf{P}\subset\Omega(\mathbf{S})$.

Suppose $G\epsilon\Omega(S)$. It follows that $G=\Omega[S_1(m_1),\ldots,S_t(m_t)]$ for finite t and $m_i\geq 0$. Let $G'=\Omega[S_1(\mathrm{sgn}(m_1)),\ldots,S_t(\mathrm{sgn}(m_t))]$ where $\mathrm{sgn}(x)=1$ if $x>0$ and $=0$ if $x=0$. Clearly $G'\leq G$ and $G\epsilon\rho^{-1}(G')$ since the S_i are nonempty. Now $G'\leq G_t=\Omega[S_1(1),\ldots,S_t(1)]$ and since **P** is monotone, $G'\epsilon$**P**. Now $G'\epsilon$**P** implies $\rho^{-1}(G')\subset$**P** and since $G\epsilon\rho^{-1}(G')$ we see that $G\epsilon$**P** proving that $\Omega(S)\subset$**P**.∎

1.2.11 Remark. The three conditions in the above theorem are logically independent of one another. We verify this by presenting three examples of classes of graphs for which two of the conditions hold, but the third fails.

1.2.12 Example: (i) fails. Let **P**$=\{K_n+K_m:n,m>0\}$. We observe that condition (i) fails since $K_1\leq 2K_1\epsilon$**P** but K_1 is not in **P**. Condition (ii) holds because if $G\epsilon$**P** we know $\rho G=2K_1$ and $\rho^{-1}(G)=\{H:\rho H=\rho G\}$. Now if $\rho H=2K_1$ it follows that H has exactly two equivalence classes modulo $\equiv$. Hence $H=K_n+K_m$ and so $\rho^{-1}(G)\subset$**P** (indeed equality holds) and (ii) is verified. Finally it is clear that the graphs $2K_1\leq 2K_2\leq 2K_3\leq\ldots$ form a composition series for **P** thus verifying (iii).

1.2.13 Example: (ii) fails. Let **P** be the class of graphs each of whose components is a path: **P**$=\{\Sigma P_{n_i}:n_i\geq 0\}$. Condition (ii) clearly fails since $K_1\epsilon$**P** and yet K_3 is not in **P**, but $K_3\epsilon\rho^{-1}(K_1)$. Condition (i) clearly holds. Finally, if we consider the sequence $P_1\leq 2P_2\leq 3P_3\leq 4P_4\leq\ldots$ we see that condition (iii) holds.

1.2.14 Example: (iii) fails. Let **P**$=$**G**$-\{G:C_4\leq G$ and $C_5\leq G\}$. In other words, **P** contains all graphs that do not have both C_4 and C_5 as induced subgraphs. We first show that (iii) fails. Suppose $G_1\leq G_2\leq G_3\leq\ldots$ is a composition series for **P**. Since $C_4\epsilon$**P** we know that $C_4\leq G_{k_1}$ for some k_1. Likewise $C_5\leq G_{k_2}$. Let $k=\max\{k_1,k_2\}$. It follows that $C_4\leq G_k$ and $C_5\leq G_k$ with $G_k\epsilon$**P**, a contradiction, hence (iii) does not hold. It is obvious that

(i) holds. To verify (ii) we observe that for $k \geq 4$, G contains an induced k-cycle if and only if ρG does also. Thus if $G \in \mathbf{P}$ and for arbitrary H satisfying $\rho H = \rho G$ we cannot have $C_4 \leq H$ and $C_5 \leq H$, hence $H \in \mathbf{P}$.

We now apply theorem 1.2.10 to some classes of graphs. In so doing, the following lemma is useful:

1.2.15 Lemma. If **P** is a class of graphs that is closed under disjoint union, then **P** has a composition series.

Proof. For each positive integer k let $\mathbf{P}_k = \{G \in \mathbf{P} : |V(G)| \leq k\}$. Clearly $\mathbf{P}_k$ is finite and let G_k be the disjoint union of the graphs in $\mathbf{P}_k$. It follows that $G_k \in \mathbf{P}$, $G_1 \leq G_2 \leq G_3 \leq \ldots$ and every graph in **P** with n vertices is an induced subgraph of G_n.∎

1.2.16 Example: Chordal Graphs. Let **P** be the class of chordal graphs (see Definition 1.1.7). It is obvious that **P** is monotone, and by the observation concerning chordless cycles at the end of Example 1.2.14 we know **P** is closed under ρ^{-1}. Finally, since the disjoint union of chordal graphs is clearly chordal, we see by Lemma 1.2.15 that **P** has a composition series. Hence by Theorem 1.2.10 there exists a family of nonempty sets **S** such that $\Omega(\mathbf{S})$ is, exactly, the class of chordal graphs. This, of course, is no surprise since by Theorem 1.1.25 we knew that the intersection graphs of subtrees of trees are the chordal graphs.

The point of this example is that although the proof of Theorem 1.2.10 is constructive, the family **S** assembled may not be the most "natural". Nor is it unique. The next few sections develop another example of an intersection family of greater interest.

1.2.17 Definition. Let $\omega(G)$ denote the size (number of vertices) of the largest clique in G and let $\chi(G)$ denote the chromatic number of G. A graph G is said to be **perfect** if for all $H \le G$ we have $\omega(H)=\chi(H)$.

The field of perfect graphs is an extremely active area of research. Lovasz has shown that this class is an intersection class of graphs by explicit construction of a family **S**.

We demonstrate nonconstructively that perfect graphs form an intersection class. We recall the following result of Berge:

1.2.18 Lemma [4]. A graph G is perfect if and only if ρG is perfect, i.e. the family of perfect graphs is closed under ρ^{-1}.∎

1.2.19 Theorem. The class of perfect graphs is an intersection class.

Proof. It is immediate from the definition that perfect graphs form a monotone class and are also closed under disjoint union. The result is now obvious in light of 1.2.15, the previous lemma and Theorem 1.2.10.∎

1.2.20 Remark. We conclude our characterization of classes $\Omega(\mathbf{S})$ by lifting the restriction that the sets in **S** be nonempty. In an **S**-representation the empty set may only be assigned to isolated vertices.

1.2.21 Definition. Let ιG be the graph formed from G by deleting its isolated vertices. Thus if G has k isolated vertices, then $G = \iota G + kK_1$.

1.2.22 Proposition. If **S** is a family of nonempty sets, then $G \in \Omega(\mathbf{S} \cup \{\emptyset\})$ if and only if $\iota G \in \Omega(\mathbf{S})$.

Proof. Suppose $G \in \Omega(\mathbf{S} \cup \{\emptyset\})$. Since $\iota G \le G$ we know $\iota G \in \Omega(\mathbf{S} \cup \{\emptyset\})$, but since no vertex of ιG is isolated, no vertex of ιG is assigned $\emptyset$ in an

$(S\cup\{\emptyset\})$-representation. Thus $\iota G\in\Omega(S)$.

Conversely, suppose $\iota G\in\Omega(S)$ and $G=\iota G+kK_1$. Thus $\iota G=\Omega[S_1(m_1),\ldots,S_n(m_n)]$ with $S_i\in S$. Clearly $G=\Omega[S_1(m_1),\ldots,S_n(m_n),\emptyset(k)]$ and therefore $G\in\Omega(S\cup\{\emptyset\})$.∎

Thus adding the empty set allows for unlimited isolated vertices.

We conclude with three examples: the first is a class of graphs representable only by nonempty sets, the second is a class representable only by families that include the empty set and the third is a class representable either way.

1.2.23 Example. Let P be the class of complete graphs: $P=\{K_n:n>0\}$. If A is any nonempty set, then $P=\Omega(\{A\})$. Clearly $P\neq\Omega(S)$ with $\emptyset\in S$ because $2K_1$ is not in P.

1.2.24 Example. Let P be the class of graphs $\{K_n+mK_1:n>0,\ m\geq 0\}$. Clearly $P=\Omega(\{A,\emptyset\})$ with A nonempty. However, P cannot be represented only by nonempty sets, since $2K_2$ is not in P but $\rho^{-1}(2K_2)=2K_1\in P$, violating 1.2.10(ii).

1.2.25 Example. Let P be the class of interval graphs. Let $S=B^1=\{[a,b]:a\leq b,\ a,b\in R\}$. Now $P=\Omega(S)$, but also $P=\Omega(S\cup\{\emptyset\})$. This is because the union of the intervals in the representation of any interval graph covers only a finite portion of R. We can then add arbitrarily many additional disjoint intervals to represent isolated vertices.

1.3 Injective Intersection Representations

In the previous section one set in S could be assigned to several vertices in a graph's representation. In this section we only allow a set of S to be used at most once. Interestingly, this restriction allows for greater flexibility in the intersection classes we can achieve.

1.3.1 Definition. Let S be a family of sets and G a graph. We say that G has an **injective S-representation** if there exists an S-representation $f:V(G)\to S$ which is an injection.

In other words, for all vertices v and w with $v\neq w$ we have:
(1) $f(v)\cap f(w)\neq\emptyset$ if and only if $v\sim w$, and
(2) $f(v)\neq f(w)$.

1.3.2 Notation. If S is a family of sets we denote the class of all graphs that have injective S-representations by $\Omega_1(S)$. Clearly $\Omega_1(S)\subset\Omega(S)$.

As in section 1.2, we wish to characterize the classes $\Omega_1(S)$ which we call **injective intersection classes**. We begin with two lemmas giving necessary conditions analogous to 1.2.1 and 1.2.9.

1.3.3 Lemma. If $P=\Omega_1(S)$ then P is monotone.

Proof. If $H\leq G\in\Omega_1(S)$ and $f:V(G)\to S$ is an injective S-representation, then so is $f|V(H)$.∎

1.3.4 Lemma. If $P=\Omega_1(S)$ then P has a composition series.

Proof. Reasoning as in §§ 1.2.6 and 1.2.7 we may assume S is

countable. Let $S=\{S_1,S_2,S_3,\ldots\}$ with $S_i \neq S_j$ for $i \neq j$. Let $G_k=\Omega[S_1,\ldots,S_k]$ Notice that $f(v_i)=S_i$, $1 \leq i \leq k$ is injective. Hence $G_1 \leq G_2 \leq G_3 \leq \ldots$ and $G_i \in P$ for all i. Clearly if G has an injective S-representation, then if $k=\max\{i: f(v)=S_i,\ v \in V(G)\}$ then $G \leq G_k$.∎

These two conditions are sufficient.

1.3.5 Theorem. Let **P** be a family of graphs. **P** is an injective intersection class if and only if the following conditions hold:

(i) **P** is monotone, and

(ii) **P** has a composition series.

Proof. The necessity of (i) and (ii) is proved in 1.3.3 and 1.3.4. Conversely, suppose **P** satisfies (i) and (ii). Let $G_1 \leq G_2 \leq G_3 \leq \ldots$ be a composition series for **P**. We may, as in the proof of Theorem 1.2.10, assume that $|V(G_i)|=i$. Let $V(G_k)=\{v_1,\ldots,v_k\}$. Let $S_i=\{(i,j):j>0\} \cup \{(j,i):j<i \text{ and } v_j \sim v_i\}$. Let $S=\{S_1,S_2,S_3,\ldots\}$. Notice that $G_k=\Omega[S_1,\ldots,S_k]$ as in 1.2.10. We show that $P=\Omega_1(S)$.

Let $G \in P$. Then $G \leq G_k=\Omega[S_1,\ldots,S_k]$ for some k. Since the S_i are all distinct (only S_i contains (i,i)) clearly $G \in \Omega_1(S)$. Thus $P \subset \Omega_1(S)$.

Let $G \in \Omega_1(S)$. Let $f:V(G) \rightarrow S$ be an injective S-representation and let $k=\max\{i: f(v)=S_i,\ v \in V(G)\}$. Clearly $G \leq G_k$ and since **P** is monotone, $G \in P$. Thus $\Omega_1(S) \subset P$.∎

1.3.6 Remark. As mentioned above, for a family of sets S we know that $\Omega_1(S) \subset \Omega(S)$. This inclusion may or may not be proper. If $S=\{A\}$ with $A \neq \emptyset$ then $\Omega_1(S)=\{K_1\}$ while $\Omega(S)=\{K_n:n>0\}$ and the inclusion is proper. However, if $S=B^1=\{[a,b]:a \leq b,\ a,b \in \mathbf{R}\}$ then $\Omega_1(S)=\Omega(S)$. To see this note that we can always slightly enlarge each interval in an interval representation by varying amounts so that all intervals in the

representation of an interval graph are distinct. [This is possible since the intervals are compact and there exists a positive number ε which is the minimum distance between nonintersecting intervals. No new intersections are formed if, in extending the intervals, the endpoints are moved less than $\frac{1}{2}\varepsilon$.]

Although $\Omega_1(\mathbf{S})\subset\Omega(\mathbf{S})$, injective representations allow for greater flexibility in forming classes of graphs. Since the conditions for an injective intersection class are a subset of those for an intersection class, we have

1.3.7 Corollary. If $\mathbf{S}$ is a family of sets and $\mathbf{P}=\Omega(\mathbf{S})$ then there exists a family of sets $\mathbf{S}'$ such that $\mathbf{P}=\Omega_1(\mathbf{S}')$.∎

Hence every intersection class of graphs is also an injective intersection class. The opposite, however, is not true, as we see in the following examples.

1.3.8 Example: Forests. If $\mathbf{P}$ is the class of forests then there exists a family $\mathbf{S}$ such that $\mathbf{P}=\Omega_1(\mathbf{S})$. We can see this by noting that $\mathbf{P}$ is clearly monotone and closed under disjoint union. The claim follows by applying 1.2.15 and 1.3.5.

If we require the components of the forests in $\mathbf{P}$ to be paths, we see, as in 1.2.13, that $\mathbf{P}=\Omega_1(\mathbf{S})$ for some $\mathbf{S}$. Indeed it is trivial to verify this is true for $\mathbf{S}=\{[a,a+1]:a\in\mathbf{Z}\}$. However, by 1.2.13, $\mathbf{P}$ does not satisfy 1.2.10(ii) and hence is not an intersection class.

1.3.9 Example: Planar Graphs. If $\mathbf{P}$ is the class of planar graphs, it is clearly monotone and closed under disjoint union. Hence, by 1.2.15 and 1.3.5 it is an injective intersection class. It is not an intersection class since K_5 is not in $\mathbf{P}$, yet $\rho K_5=K_1\in\mathbf{P}$, violating

1 2.10(ii).

1.3.10 Example: Line Graphs. If G is a graph, then L(G) is the intersection graph of the edges of G. Clearly the family of all line graphs is an injective intersection family. The class of all line graphs is not an intersection class because it is not closed under ρ^{-1} as the following example shows: Observe that $L(C_4)=C_4$ and hence C_4 is a line graph. Now let G be the graph shown in figure 1.6. It is known that

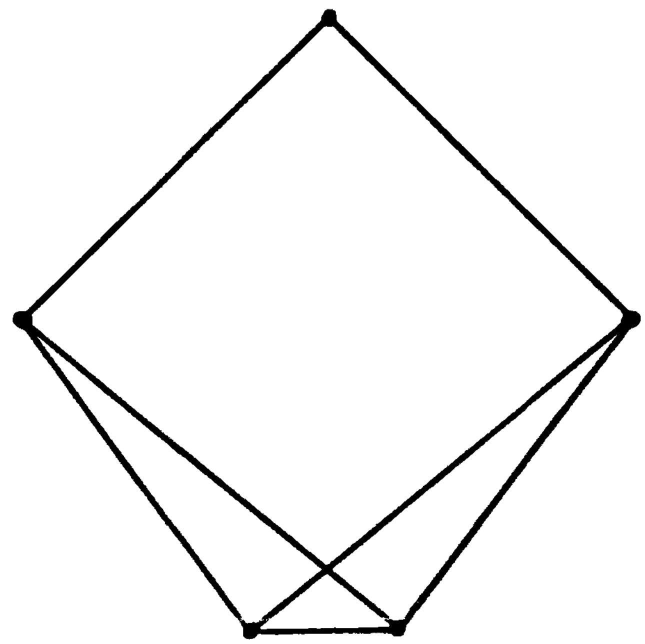


Figure 1.6

G is not a line graph [1], however, $\rho G=C_4$.

1.4 Nerves

In this section we discuss an analogue of intersection graph: the nerve of a collection of sets. This will extend the results of the previous section to simplicial complexes. We prove a characterization theorem for families of nerves analogous to Theorem 1.3.5.

1.4.1 Definition. Let S be a finite family of nonempty sets. The **nerve** of S, denoted N(S), is a hypergraph (V,E) for which there exists a bijection $f:V\longrightarrow S$ such that $\{v_1,\ldots,v_k\}\in E$ if and only if $f(v_1)\cap\ldots\cap f(v_k)\neq\emptyset$.

1.4.2 Remark. Under our definition, the nerve is defined only for finite collections of nonempty sets. Some authors define nerves for arbitrary collections, but this can result in infinite hypergraphs, which we do not discuss. The nonemptiness requirement is a simplifying assumption; the empty set would only serve to add an isolated vertex.

It is also possible to define the nerve for a multiset of sets, but we will not need this generality.

Finally, the definition actually only defines the nerve up to hypergraph isomorphism. This will not cause a problem as we generally identify isomorphic hypergraphs. One could assume that the vertex set of N(S) is the collection S itself and the edges of N(S) to be those subsets of S whose elements have nonempty intersection. In this way the nerve would be uniquely defined.

The following is immediate from the definitions:

1.4.3 Proposition. The nerve of a family is a simplicial complex.∎

1.4.4 Definition. Let $\mathbf{S}$ be a collection of nonempty sets. Let $\mathbf{N}(\mathbf{S})$ denote the class of nerves of finite subsets of $\mathbf{S}$, i.e.,

$$\mathbf{N}(\mathbf{S})=\{N(S):S\subset\mathbf{S} \text{ and } |S|<\infty\}.$$

We call such a class of simplicial complexes a **nerve class**.

Note that the 1-skeleton of a nerve of a family can be identified with the intersection graph of the sets of that family and that $\Omega_1=sk_1\circ\mathbf{N}$.

We now characterize nerve classes.

1.4.5 Theorem. If $\mathbf{P}$ is a family of simplicial complexes, then $\mathbf{P}$ is a nerve class if and only if the following conditions hold:

(i) $\mathbf{P}$ is monotone (i.e. is closed under induced subcomplexes), and

(ii) $\mathbf{P}$ has a composition series: $K_1\le K_2\le K_3\le\ldots$ with $K_i\in\mathbf{P}$ such that if $K\in\mathbf{P}$ then $K\le K_i$ for some i.

Proof. It is routine to verify the necessity of (i) and (ii). Suppose $\mathbf{P}$ satisfies conditions (i) and (ii). Since $\mathbf{P}$ is monotone we refine its composition series such that $|V(K_k)|=k$, and that $V(K_k)=\{v_1,\ldots,v_k\}$. We now define sets S_i whose elements will be finite subsets of $\mathbf{Z}^+$. Let

$$S_i = \{A\subset\mathbf{Z}^+: i\in A \text{ and } \{v_j:j\in A\}\in E(K_k) \text{ for some } k\}.$$

Let $S=\{S_1,S_2,S_3,\ldots\}$. We claim that $\mathbf{P}=\mathbf{N}(S)$. This will be immediate once we verify that $K_k=N(\{S_1,\ldots,S_k\})$. Let $f(v_i)=S_i$ for $1\le i\le k$. Let $\{v_{i_1},\ldots,v_{i_p}\}$ be an edge of K_k. It follows at once that $\{i_1,\ldots,i_p\}\in S_{i_1}\cap\ldots\cap S_{i_p}$. Conversely, if $A\in S_{i_1}\cap\ldots\cap S_{i_p}$ we know that $A\in S_j$ implies $j\in A$, hence $A\supset\{i_1,\ldots,i_p\}$ and therefore $\{v_{i_1},\ldots,v_{i_p}\}$ is an edge. Hence $K_k=N(\{S_1,\ldots,S_k\})$.

Now if $K \in \mathbf{P}$ we know by (ii) that $K \leq K_k \in \mathbf{N}(\mathbf{S})$, thus $K \in \mathbf{N}(\mathbf{S})$ and hence $\mathbf{P} \subset \mathbf{N}(\mathbf{S})$. If $K \in \mathbf{N}(\mathbf{S})$, then $K = N(\{S_{i_1}, \ldots, S_{i_p}\})$. Let $k = \max\{i_1, \ldots, i_p\}$. It follows that $K \leq N(\{S_1, \ldots, S_k\}) = K_k \in \mathbf{P}$. We know then by (i) that $K \in \mathbf{P}$, so $\mathbf{N}(\mathbf{S}) \subset \mathbf{P}$.∎

1.4.6 Remark. By taking 1-skeletons, Theorem 1.3.5 follows from 1.4.5 as a corollary.

The proofs of all three characterization theorems (1.2.10, 1.3.5 and 1.4.5) are constructive in nature, however, this last proof is "less constructive" than the other two. In the proofs of 1.2.10 and 1.3.5 the sets S_i can be constructed just by knowing the graphs $G_1, \ldots, G_i$. However, to construct an S_i in the proof of 1.4.5 we needed an "infinite amount of information"; we required "knowledge" of the entire class $\mathbf{P}$. This can be avoided if we define the sets S_i so that $|S_{i_1} \cap \ldots \cap S_{i_p}| = 0$ or ∞. Then when a new S_j is defined we make sure that when S_j must meet $S_{i_1} \cap \ldots \cap S_{i_p}$ that both $S_j \cap (S_{i_1} \cap \ldots \cap S_{i_p})$ and $(S_{i_1} \cap \ldots \cap S_{i_p}) - S_j$ are infinite. This will always allow for enough flexibility as the next K_k is examined we can form exactly whatever edges we wish. Such a proof, in which each S_k is defined only in terms of $K_1, \ldots, K_k$ is therefore possible, but would be quite tedious.

We now present an alternative characterization of nerve classes in which we allow our simplicial complexes to be infinite. A **countable** simplicial complex is one in which the vertex set is countable.

1.4.7 Corollary. A class $\mathbf{P}$ of simplicial complexes is a nerve class if and only if there exists a countable simplicial complex L such that $\mathbf{P} = \{K : K \leq L, K \text{ a finite simplicial complex}\}$.

Proof. Let $\mathbf{P}$ be a nerve class, $\mathbf{P} = \mathbf{N}(\mathbf{S})$ for some collection $\mathbf{S}$ of nonempty sets. As in 1.2.7 we may assume that $\mathbf{S}$ is countable. Let

$L=N(\mathbf{S})$, i.e. the nerve of the countable collection $\mathbf{S}$, a countable simplicial complex. Clearly $\mathbf{P}=\{K:K\leq L, K \text{ finite}\}$.

Conversely, let $\mathbf{P}=\{K:K\leq L, K \text{ finite}\}$ for some countable simplicial complex L. If $K\in\mathbf{P}$ and $K'\leq K$ then since $K\leq L$, we have $K'\leq L$ and $K'\in\mathbf{P}$. Hence $\mathbf{P}$ is monotone.

Let $V(L) = \{v_1,v_2,v_3,\ldots\}$. Let K_k be the (finite) induced subcomplex of L on vertices $\{v_1,\ldots,v_k\}$. Clearly $K_1\leq K_2\leq K_3\leq\ldots$ and if $K\in\mathbf{P}$ then for $k=\max\{k:v_k\in V(K)\}$ we have $K\leq K_k$. Thus the K_i form a composition series and by 1.4.5 $\mathbf{P}$ is a nerve class.∎

A similar corollary can be proved for Theorem 1.3.5 involving infinite graphs.

1.4.8 Example: Convex Sets. Recall that $\mathbf{K}^d$ denotes the set of all compact convex sets in $\mathbf{R}^d$ (see §1.1.21). The class $N(\mathbf{K}^1)$ consists exactly of those simplicial complexes K whose 1-skeletons (a) are interval graphs and (b) whose cliques correspond to edges (simplices) of K. This is an immediate consequence of the Helly property for intervals: every family of real intervals which intersect pairwise have nonempty intersection.

The classes $N(\mathbf{K}^d)$ are discussed in [77]. The primary result in which we are interested is the following:

1.4.9 Theorem [77]. The class $N(\mathbf{K}^{2k+1})$ contains all k-dimensional simplicial complexes for every positive integer k.∎

1.5 Computational Complexity

Let **P** be an intersection class of graphs. We are interested in determining the difficulty of computing when a given graph is an element of **P**. For example, if $\mathbf{P}=\Omega(2^{\mathbf{Z}})$ it is trivial (constant time) to determine membership in **P**. (The answer is always "yes".) If **P** is the class of interval graphs, then determination of a graph's membership in **P** can be accomplished in linear time, while if $\mathbf{P}=\Omega(\mathbf{B}^3)$ the corresponding membership problem is NP-complete [80]. What can be said, in general, about the computational complexity of membership in an intersection class? Are all such decision problems in the class NP? Are they all even decidable? We shall see that there exist intersection classes for which the time complexity of testing membership is polynomially equivalent to an arbitrary decision problem. Indeed, there are intersection classes for which membership is undecidable. We begin with such an example.

1.5.1 Proposition. Let A be a subset of the integers each of whose members is at least 4. If

$$\mathbf{P} = \{G\colon\ \rho G \le C_{n_1}+\ldots+C_{n_t} \text{ with } n_i \in A\}$$

then **P** is an intersection class and (for $n \ge 4$) $C_n \in \mathbf{P}$ if and only if $n \in A$.

Proof. It is clear that **P** is monotone and closed under ρ^{-1}. Since **P** is closed under disjoint union, 1.2.15 implies **P** has a composition series and it follows from 1.2.10 that **P** is an intersection class.

Suppose $n \in A$. Since $n \ge 4$ and $\rho C_n = C_n$, it follows that $C_n \in \mathbf{P}$. Conversely, if $n \ge 4$ and $C_n \in \mathbf{P}$ then $C_n = \rho C_n \le C_{n_1}+\ldots+C_{n_t}$ with $n_i \in A$. Thus $C_n \le C_{n_j}$ for some j, implying $C_n = C_{n_j}$ and $n = n_j \in A$. ∎

1.5.2 Theorem. There exist intersection classes for which testing membership is undecidable.

Proof. There exist sets of integers for which determining membership is undecidable. Let A be such a set with the condition that all its elements are at least 4. Let P be as in the above proposition. It is now clear that determining $C_n \in P$ is undecidable and hence membership in general is undecidable.∎

1.5.3 Remark. The structure of the set A is "encoded" in the structure of the class P by the use of cycles. This encoding scheme is sufficient to prove undecidability, but is not of much help in other computational complexity work. This is because the encoding of an integer n is essentially in unary requiring exponentially more "bits" than the log n typically required. The transformation is not polynomial. Stefan Burr [personal communication] recommended encoding integers by using

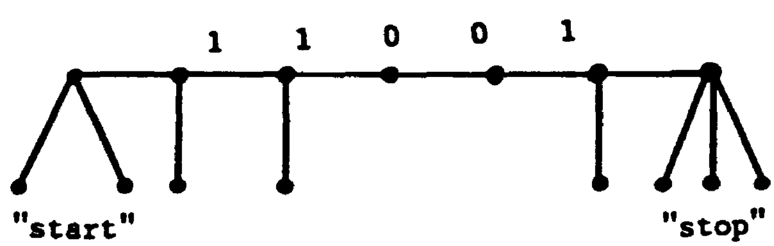


Figure 1.7

"caterpillars"; see Figure 1.7. This method can encode bit strings in a polynomial (actually linear) fashion, but is not exactly what we need for encoding in an intersection class because such classes are monotone. Thus, once a bit string is encoded we may be forced to include all strings which are "bit by bit less than" the one encoded. A modification of Burr's suggestion is presented here which will accomplish our

objectives.

1.5.4 Definition. Let $k_1,k_2,\dots,k_t$ be integers with each $k_i=0$ or $k_i\geq 4$. We define a graph $Q(k_1,\dots,k_t)$ to consist of a path on t vertices: $v_1\sim v_2\sim\dots\sim v_t$ and for each i a cycle on k_i vertices: $v_i=v_{i,0}\sim v_{i,1}\sim\dots\sim v_{i,k_i-1}\sim v_{i,0}$ with all $v_{i,j}$ distinct. (In case $k_i=0$

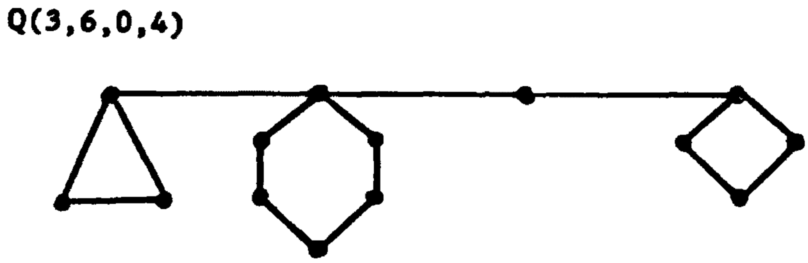
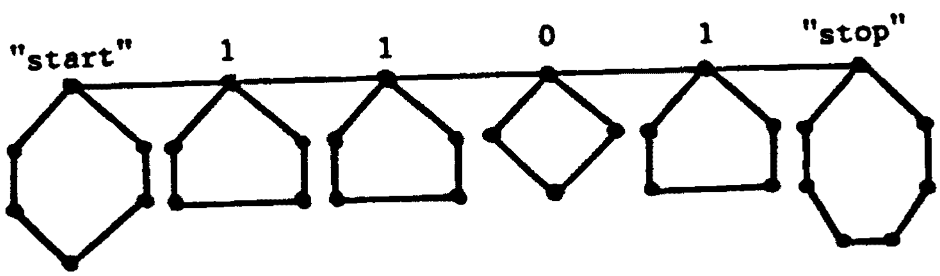


Figure 1.8

there is no cycle.) See figure 1.8. Such a graph is called a **Q-graph**.

1.5.5 Definition. Let n be a positive integer and $n=\Sigma_1^t a_i 2^{t-i}$ is a representation of n in binary: $a_1=1$ and $a_i=0,1$ $(i=2,\dots,t)$. Put $k_i=a_i+4$.



$\Gamma(13) = \Gamma(1101_{two}) = Q(6,5,5,4,5,7)$

Figure 1.9

Define the graph $\Gamma(n)$ to be $Q(6,k_1,k_2,\dots,k_t,7)$. See figure 1.9.

1.5.6 Definition. Let G be a graph. We define the **pruned** subgraph of G, denoted πG, recursively as follows:

(1) If G has no vertices of degree 1, then $\pi G=G$, otherwise

(2) If $v\in V(G)$ with $d_G(v) = 1$ then $\pi G = \pi(G-v)$.

Thus πG is formed by iteratively deleting vertices of degree 1 from G. It is a routine exercise to verify that it does not matter in what order vertices of degree 1 are removed; therefore, π is well-defined.

1.5.7 Proposition. If G is a Q-graph and H is connected with $H\le G$, then πH is a Q-graph.

Proof. If H is a tree, then $\pi H=K_1=Q(0)$. If H has exactly one cycle, then $\pi H=C_k=Q(k)$. Let $H\le G=Q(k_1,\ldots,k_t)$ be connected with at least two cycles. Let $x_1\sim x_2\sim\ldots\sim x_t$ be the "main path" in G. We may assume some of the x_i's are in H (otherwise H is acyclic) and those in H are consecutive (otherwise H would not be connected): $x_j\sim x_{j+1}\sim\ldots\sim x_s$, $1\le j<s\le t$. In πH we have the following "main" vertices: $x_p\sim\ldots\sim x_q$ with $1\le j\le p<q\le s\le t$. (We have $j<p$, for example, in case the k_j-cycle attached at x_j is not complete in H.) We may assume that the k_p and k_q cycles at x_p and x_q are complete in H. For r with $p<r<q$ let $n_r=k_r$ if all vertices of the k_r-cycle at x_r are in H and $n_r=0$ otherwise. If follows that $\pi H=Q(k_p,n_{p+1},\ldots,n_{q-1},k_q)$.█

1.5.8 Definition. Let A be a set of positive integers. We define $P(A)$ to be the set of all graphs G each of whose connected components H satisfy either:

(1) $\pi\rho H=\Gamma(n)$ for some $n\in A$, or

(2) $\pi\rho H<\Gamma(n)$ for some n.

1.5.9 Proposition. If $A\subset Z^+$ then $\mathbf{P}(A)$ is an intersection class.

Proof. Clearly $\mathbf{P}=\mathbf{P}(A)$ is closed under disjoint union and hence possesses a composition series. Note that $\pi\rho(\rho H)=\pi\rho H$ and if $G=H_1+\ldots+H_t$ then $\rho G=\rho H_1+\ldots+\rho H_t$. It follows that $\rho G\in\mathbf{P}$ implies $G\in\mathbf{P}$, hence $\mathbf{P}$ is closed under ρ^{-1}. It remains to show that $\mathbf{P}$ is monotone.

Suppose G is connected, $G\in\mathbf{P}$ and $H\leq G$. If $H=G$ we are done. Suppose $H<G$. Since $G\leq\Gamma(n)$ for some n, we have $\pi\rho H\leq\rho H\leq H<G\leq\Gamma(n)$ hence H, whether connected or not, is also in $\mathbf{P}$. If G is not connected, then this exact argument can be applied to each of its components and we conclude that $\mathbf{P}$ is an intersection class.∎

1.5.10 Lemma. If $H\leq G$ then $\rho H\leq\rho G$ and $\pi H\leq\pi G$.

Proof. If $H\leq G$ and x and y are equivalent vertices of G in V(H) then clearly x and y are equivalent in H. Hence $\rho H\leq\rho G$.

Next suppose $v\in V(H)$. Clearly $d_H(v)\leq d_G(v)$ and thus if $d_G(v)=1$ then either $d_H(v)=1$ or $d_H(v)=0$. In the first case we note that v would be removed from H as well and in the second case, since πH has the same number of connected components as H, no problem can arise.∎

1.5.11 Theorem. If $A\subset Z^+$ then $\Gamma(n)\in\mathbf{P}(A)$ if and only if $n\in A$.

Proof. If $n\in A$ it is immediate that $\Gamma(n)\in\mathbf{P}(A)$. Suppose $\Gamma(n)\in\mathbf{P}$. Since $\Gamma(n)$ is connected we know that $\pi\rho\Gamma(n)=\Gamma(n')$ with $n'\in A$ or else $\pi\rho\Gamma(n)<\Gamma(n')$ with n' arbitrary. In the first case since $\pi\rho\Gamma(n)=\Gamma(n)$ we have $\Gamma(n)=\Gamma(n')$. Clearly $n=n'$ and hence $n\in A$. Suppose $\Gamma(n)<\Gamma(n')$. Both $\Gamma(n)$ and $\Gamma(n')$ have a 6- and 7-cycle which must correspond. The connecting path between these cycles must be of the same length. Next, in each position along the path we must have $k_i=k_i'$ since different size

cycles cannot be one an induced subgraph of the other. Thus $\Gamma(n)=\Gamma(n')$ contradicting $\Gamma(n)<\Gamma(n')$. Thus $n\in A$.▮

1.5.12 Definition. Let $A\subset Z^+$ and $\mathbf{P}=\mathbf{P}(A)$ be fixed. We define two decision problems:

- Problem $\Pi_{\mathbf{P}}$: Instance: a graph G. Question: Is $G\in\mathbf{P}$?
- Problem Π_A: Instance: a finite multiset of integers M. Question: Is every element of M in $A\cup\{0\}$?

1.5.13 Theorem. $\Pi_A \propto \Pi_{\mathbf{P}}$. [Problem Π_A is transformable via a polynomial time deterministic algorithm to problem $\Pi_{\mathbf{P}}$.]

Proof. We must exhibit a function (computable in polynomial time) mapping every instance of Π_A to an instance of $\Pi_{\mathbf{P}}$ with the property that accepting instances correspond. Let M be a multiset of integers: $M=[n_1,\ldots,n_k]$. Let G(M) be the graph with $k=|M|$ connected components $H_1,\ldots,H_k$ with H_i defined by:

$$\begin{aligned}&W_5 = C_5\wedge K_1 \text{ if } n_i<0,\\ &K_1 \text{ if } n=0, \text{ or}\\ &\Gamma(n_i), \text{ if } n_i>0.\end{aligned}$$

Given M, it is clear that we can compute G(M), an instance of $\Pi_{\mathbf{P}}$, in polynomial time. Suppose $M\subset A\cup\{0\}$. Then each element of M is either 0 or in A and the corresponding component H of G(M) is either $K_1\in\mathbf{P}$ or $\Gamma(n)$ for $n\in A$ implying $\Gamma(n)\in\mathbf{P}$ by 1.5.12. Since $\mathbf{P}$ is closed under disjoint union, $G(M)\in\mathbf{P}$.

Conversely, suppose $G(M)\in\mathbf{P}$ for some finite multiset M of integers. If $W_5\le G(M)$ then G(M) is not in $\mathbf{P}$ since W_5 is not. Hence all members of M are non-negative. Let $x\in M$. If $x\neq 0$ then $x>0$. Thus $\Gamma(x)$ is a component of G(M) and thus $\Gamma(x)\in\mathbf{P}$ and so $x\in A$ by 1.5.12. If follows that $M\subset A\cup\{0\}$. Thus $M\longmapsto G(M)$ is a polynomial time map with M an accepting input to Π_A if and only if G(M) is an accepting input to $\Pi_{\mathbf{P}}$.▮

1.5.14 Theorem. $\Pi_P = \Pi_A$.

Proof. Let G be a graph. We construct a multiset M(G) as follows: If H is a connected graph let f(H) be defined by f(H) =

n if $\pi\rho H = \Gamma(n)$ for some n,

0 if $\pi\rho H < \Gamma(n)$ for some n, or

-1 otherwise.

Now suppose G has connected components $H_1,\ldots,H_k$. Put $M(G)=[f(H_1),f(H_2).,\ldots,f(H_k)]$. We claim that M(G) is computable in polynomial time. Since the identification of a graph's connected components can be accomplished in polynomial time, we need only verify that f can be computed in polynomial time.

To begin we note that the function π can easily be computed in polynomial time—its definition is the algorithm! Likewise we can compute ρG in polynomial time as follows:

Algorithm: RHO

Input: Graph G=(V,E)

Output: ρG

(1) We assume G's vertices are $V=\{v_1,v_2,\ldots,v_n\}$. Let $E'=\emptyset$.

(2) FOR i=1 to n DO:

 (2a) FOR j=i+1 to n DO:

  (2a') IF $adj(v_i)\cup\{v_i\} = adj(v_j)\cup\{v_j\}$ THEN

   put $E'\leftarrow E'\cup\{v_i v_j\}$.

 (2b) NEXT j.

(3) NEXT i.

(4) Choose one vertex from each connected component of (V,E') and output the induced subgraph of G on these vertices.

It is immediate that RHO runs in polynomial time. The output is clearly ρG since components of (V,E') are exactly the $\equiv$ equivalence classes.

Next we show how to recognize a Q-graph in polynomial time.

Algorithm: QTEST

Input: Connected Graph G

Output: sequence of integers $k_1,\ldots,k_t$ with $G=Q(k_1,\ldots,k_t)$ or "REJECT" if no such sequence exists.

[Note: a "REJECT" instruction below means to output "REJECT" and halt.]

(1) If $\Delta(G)>4$ then REJECT.

(2) Find all vertices of degree 3. There should be exactly two. If so call them x and x'. If not, REJECT.

(3) Find a shortest path from x to x'. Let t be its length. Let $x=x_1\sim x_2\sim\ldots\sim x_t=x'$ be this path. Call all edges x_ix_{i+1} "forbidden".

(4) For i=1 to t DO:

(4a) Find all the vertices reachable from x_i by non-forbidden edges. If none other than x_i, put $k_i=0$ and go to step (5).

(4b) If any of these "reachable" vertices is another x_j $(j\neq i)$ then REJECT.

(4c) If these "reachable" vertices (together with x_i) form a k-cycle then let $k_i=k$, otherwise REJECT.

(5) Next i.

(6) Output the sequence $k_1,\ldots,k_t$.

The correctness and polynomial time complexity of the above algorithm are readily verifiable.

We now give an algorithm to compute f(H) for connected H in polynomial time.

Algorithm: F

Input: Connected graph H

Output: f(H), as defined above

(1) Let $H'=\pi\rho H$.

(2) Process H' using QTEST. If rejected, then output (-1) and halt. Otherwise, let $k_1,\ldots,k_t$ be the output of algorithm QTEST.

(3) If any $k_i \geq 8$ OR if any $k_i \geq 6$ with $1<i<t$ then output (-1) and halt. If any $k_i=3$ then output (-1) and halt.

(4) If (k_1,k_t) equals ...

a) (6,6) output (-1) and halt,

b) (7,7) output (-1) and halt,

c) (7,6) "reverse" the string of k's.

(5) If $(k_1,k_2,k_t)\neq(6,5,7)$ then output (0) and halt.

(6) Let N=0.

(7) For j=2 to t-1 DO:

(7a) If $k_j=4$ then let N=2N.

(7b) If $k_j=5$ then let N=2N+1.

(7c) If $k_j\neq 4,5$ then output (0) and halt.

(8) Next j.

(9) Output (N).

Clearly the algorithm runs in polynomial time. It is routine to verify the correctness of this algorithm. Step 2 weeds out non-Q-graphs. Steps 3 and 4 weed out graphs which cannot be a subgraph of a Γ-graph. In step 5 we eliminate the "obvious" non-Γ-graphs. Steps 6 through 8 calculate N in case $H'=\Gamma(N)$ or output 0 in case H' is not a Γ-graph, but a sub-Γ-graph. Thus the function f is computable in polynomial time.

It remains to show that G is an accepting instance of $\Pi_{\mathbf{P}}$ if and only if M(G) is an accepting instance of Π_A. Let $G=H_1+\ldots+H_t$ with each H_i connected. Suppose $G\in\mathbf{P}$. Then $f(H_i)\neq-1$ by definition of $\mathbf{P}$. If $f(H_i)=n>0$, then $\pi\rho H_i=\Gamma(n)$ and $n\in A$. Thus $f(H_i)\in A\cup\{0\}$ for all i. Thus

$M(G)$ is an accepting input for Π_A.

Conversely, suppose $M(G)$ is an accepting input for Π_A. Then we see that $f(H_i) \in M(G) \subset A \cup \{0\}$. If $f(H_i)=0$ then clearly $H_i \in P$. Similarly, if $f(H_i)=n$ then $\pi\rho H_i=\Gamma(n)$ and since $n \in A$ we know that $H_i \in P$. Thus $G(M) \in P$ and is therefore an accepting input for Π_P.∎

1.5.15 Corollary. The (time) computational complexity of testing membership in an intersection class is polynomially equivalent to any arbitrary decision problem.

Proof. By 1.5.13 and 1.5.14 Π_P and Π_A are polynomially equivalent. Every decision problem can be expressed in the form "is $x \in A$?" for some set of positive integers A. [Every instance can be encoded as a bit string which we may assume begins with a "1".] The computational complexity of Π_A is polynomially equivalent to testing $x \in A$, since we can check each member of M serially for membership in $A \cup \{0\}$ accepting exactly in case all the individuals are accepted.∎

1.5.16 Remark. Since every intersection class is also an injective intersection class, it follows that the above result applies to injective intersection classes as well. Similarly, we can consider P(A) to be a nerve class and extend the above to nerve classes.

2. INTERVAL NUMBER

In the late 1950's the question was raised: What are the intersection graphs of real intervals (see [34])? Nathematicians were calling for a characterization of interval graphs. At about the same time, biologists were investigating the workings of heredity. It was understood that genes appeared sequentially on chromosomes, but their fine structure was unknown. Benzer [2] utilized the new graph theoretic concept of interval graph to assertain that the viral genes he was studying were linear. His work initiated the concept that genes were **intervals** on the genetic strand.

This model went unchallenged until the late 1970's when Chambon [9] and others discovered genes which did not fit into this framework. They demonstrated that some genes were not single intervals on the DNA strand, but rather consisted of several disjoint intervals. A single gene may be the of a union of finitely many disjoint segments of DNA. Coincidentally in the mathematics community, researchers were generalizing the concept of interval graph to multiple interval graphs; graphs which are the intersection graphs of sets which are unions of intervals! (See [30] and [68].)

In this chapter we will formally introduce the concept of multiple interval graphs and the interval number.

A **double interval** is the union of two real intervals and a **double interval graph** is a graph which posseses an intersection representation by double intervals. Similarly, one can define a **t-interval** as the union of t real intervals and their intersection graphs are called **t-interval graphs.** The most fruitful concept in studying multiple interval graphs is the **interval number** of a graph which is the least t for which the given graph is a t-interval graph.

In this chapter we begin discuss the computation of the interval number and its relationship to other graph theoretic concepts.

2.1 Computing the Interval Number

In this section we formally introduce the interval number parameter and some of its relatives. We discuss its value for various classes of graphs.

2.1.1 Definition. Let $\mathbf{S}$ be a family of sets and let t be a positive integer. The family $t\mathbf{S}$ is defined to be the family of all t-fold unions of sets in $\mathbf{S}$, i.e.,

$$t\mathbf{S} = \{S_1 \cup \ldots \cup S_t : S_i \in \mathbf{S}\}.$$

Observe that $1\mathbf{S}=\mathbf{S}$ and that $1\mathbf{S} \subset 2\mathbf{S} \subset 3\mathbf{S} \subset \ldots$ since in the definition of $t\mathbf{S}$ the S_i need not be distinct.

2.1.2 Definition. Let $\mathbf{S}$ be a family of sets and let G be a graph. The **S-number** of G, denoted $\mathbf{S}\#(G)$, is the least positive integer t such that $G \in \Omega(t\mathbf{S})$. If no such t exists we set $\mathbf{S}\#(G)=\infty$.

2.1.3 Remark. We saw in chapter 1 (see, e.g., §1.2.16) that an intersection class of graphs may be represented by many different families of sets. We show in chapter 4 that if $\Omega(\mathbf{S})=\Omega(\mathbf{S}')$, then for all positive integers t we have $\Omega(t\mathbf{S})=\Omega(t\mathbf{S}')$. (See §4.2.4.) Thus one can define $\mathbf{S}\#(G)$ purely in terms of the family $\Omega(\mathbf{S})$. We investigate this approach in chapter 4.

2.1.4 Proposition. If $\mathbf{S}$ is a family of sets and G is graph, then $\mathbf{S}\#(G)=1$ if and only if $G \in \Omega(\mathbf{S})$.∎

2.1.5 Definition. Let S be the family of real intervals, $S=B^1$. For a positive integer t, a **t-interval graph** G is a member of $\Omega(tS)$ and $f:V(G)\rightarrow tS$ is a **t-interval representation** for G. The **interval number** of a graph G is denoted i(G) and is given by

$$i(G) = S\#(G) = \inf\{t:\ G\in\Omega(tS)\}.$$

2.1.6 Remark. The interval number of a graph is always finite. Unfortunately, it is, in general, difficult to actually compute the interval number of a given graph. For any graph G the determination whether $i(G)=1$ is easy [7], but in all other cases computation of i(G) is NP-hard due to the following result of West and Shmoys:

2.1.7 Theorem [79]. For every integer $k\geq 2$ the determination whether $i(G)\leq k$ is NP-complete.█

2.1.8 Remark. It follows that a polynomial time algorithm for the computation of the interval number is unlikely to exist. Instead we can compute the interval number for certain families of graphs. For example $i(K_n)=1$ trivially because complete graphs are interval graphs. The next easiest result concerns the interval number of trees.

2.1.9 Definition. A tree T is called a **caterpillar** if after removing the vertices of T of degree 1, the remaining vertices form a path.

2.1.10 Theorem [68]. If T is a tree then $i(T)=1$ if T is a caterpillar and $i(T)=2$ otherwise.█

Next we consider complete bipartite graphs.

2.1.11 Theorem [68]. The interval number of K(n,m) is given by

$$i(K(n,m)) = \lceil (nm+1)/(n+m) \rceil.$$ █

Finally, the most impressive result along theses lines is the computation of the interval number for complete multipartite graphs by Hopkins, Trotter and West:

2.1.12 Theorem [40]. Let $n_1 \geq n_2 \geq \ldots \geq n_p$ be positive integers. Then

$$i(K(n_1,n_2,\ldots,n_p)) = i(K(n_1,n_2))$$

most of the time, however if $(n_1,n_2)=(7,5)$ or if $(n_1,n_2)=(q^2-q-1,q)$ with $q \geq 3$ then one has

$$i(K(n_1,n_2,\ldots,n_p)) \leq i(K(n_1,n_2))+1$$

and this inequality is sharp.∎

2.1.13 Remark. Because of the intractable nature of the interval number, researchers have studies parameters closely related to interval number. Two such parameters that have been useful include the displayed interval number and the depth-r interval number which we now define.

2.1.14 Definition. A t-interval representation f of a graph G is called **displayed** if for each $v \in V(G)$ there is a non-empty open set $O_v \subset f(v)$ so that for $w \neq v$ we have $f(w) \cap O_v = \emptyset$. The set O_v is the **displayed portion** of f(v). The least positive integer t for which a graph G has a displayed t-interval representation is called the **displayed interval number** of G and is denoted $i^+(G)$.

The displayed interval number is very closely related to the interval number but is somewhat easier with which to work in inductive proofs. We present, after a definition, the main relationships between i and i^+.

2.1.15 Definition. Let G be a graph with vertices $v_1,v_2,\ldots,v_n$. We define the graph G^+ by adjoining n new vertices $w_1,w_2,\ldots,w_n$ to G so that $v_i \sim w_i$ for $i=1,\ldots,n$. See figure 2.1.

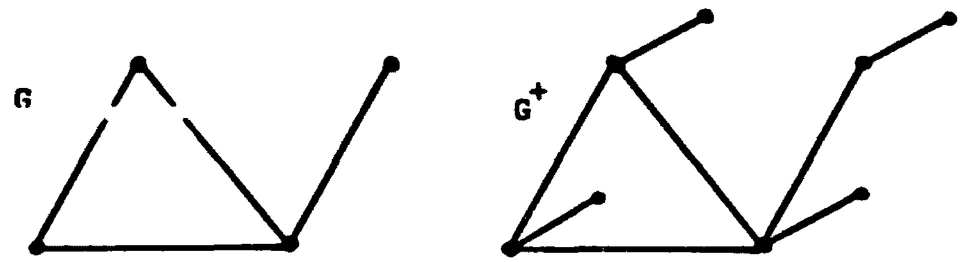


Figure 2.1

2.1.16 Theorem [62]. For all graphs, G, we have:

$$i^+(G) = i(G^+)$$

and

$$i(G) \le i^+(G) \le i(G)+1.$$

Proof. Given a displayed $i^+(G)$-interval representation for G, assign to each added vertex in G^+ an interval contained in the displayed portion of its (unique) neighbor in G. This gives an $i^+(G)$-interval representation for G^+. Thus $i(G^+) \le i^+(G)$. Now given an $i(G^+)$-interval representation for G^+ we delete the intervals for the leaves in G^+ giving a displayed $i(G^+)$-interval representation for G. Thus $i(G^+) \ge i^+(G)$.

For the second relationship we note that $G \le H$ obviously implies $i(G) \le i(H)$ and thus $i(G) \le i(G^+) = i^+(G)$. Finally, if $i(G)=t$, by adding an extra disjoint interval for each vertex in G we can form a displayed $(t+1)$-interval representation for G. Hence $i^+(G) \le i(G)+1$. ∎

Thus precise information about i^+ gives good information about i.

2.1.17 Definition. A t-interval representation of a graph G is said to have **depth-r**, where r is an integer, provided for every point x on the real line there are at most r vertices v in G for which $x \in f(v)$.

The least integer t for which G has a depth-r t-interval representation is called the **depth-r interval number** of G and is denoted $i_r(G)$. In case we require the representation to be displayed as well we have $i_r^+(G)$.

Of the depth-r parameters, i_2 is the most useful because it is the easiest to estimate. Indeed for integers $r \geq 3$ and $k \geq 2$ West and Shmoys' proof [79] shows that determining if $i_r(G) \leq k$ is NP-complete. Depth-1, although well defined, is a vacuous concept because we have $i_1(G)=1$ in case G consists of only isolated vertices, and $i_1(G)=\infty$ otherwise.

2.1.18 Theorem. For any graph G and any integer r we have,

$$i_r(G) \geq i_{r+1}(G) \geq i(G),$$
$$i_r^+(G) = i_r(G^+), \text{ and}$$
$$i_r(G) \leq i_r^+(G) \leq i_r(G)+1.$$

Proof. The first result is immediate since depth-r implies depth-(r+1). The other two results follow by analogy to 2.1.16.∎

2.1.19 Theorem. If G is a graph whose maximum clique is of size $\omega(G) \leq r$, then $i_r(G)=i(G)$.

Proof. Since $\omega(G) \leq r$ any t-interval representation of G is necessarily depth-r because if some point $x \in \mathbb{R}$ were contained in f(v) for r+1 vertices v those vertices would necessarily form a clique of size r+1.∎

2.1.20 Theorem. If G is a graph, then $i_2(G) \le (\omega(G)-1)i(G)$.

Proof. Let $r=\omega(G)$, so $i_r(G)=i(G)$. Form a depth-r i(G)-interval representation for G. One can imagine a drawing of such a representation as occupying r "layers" in which each interval lies in one of the layers and meets no other interval in its layer. Call the layers $L_1, L_2, \ldots, L_r$. Now all edges represented by edges between a pair of layers can be represented in a depth-2 fashion by "recopying" the two layers in an unused portion of the line. If we do this in all $\binom{r}{2}$ possilbe ways, we arrive at a depth-2 representation for G. One checks that each layer is recopied exactly r-1 times.∎

2.2 Interval Number Versus Graph Theory

In this section we discuss the relationship between the interval number of a graph and other graph theoretic notions.

2.2.1 Theorem [29]. If G is a graph with $n=|V(G)|$ then

$$i(G) \le \left\lceil \tfrac{1}{4}(n+1) \right\rceil$$

and this inequality is sharp.∎

2.2.2 Theorem [30]. If G is a graph with $m=|E(G)|$, then

$$i(G) \le \sqrt{m}.$$∎

2.2.3 Theorem [29]. If G is a graph with maximum degree $\Delta=\Delta(G)$, then

$$i_2(G) \le \left\lceil \tfrac{1}{2}(\Delta+1) \right\rceil.$$

Equality holds in case G is regular.∎

2.2.4 Theorem [30]. Let G be a graph with $n=|V(G)|$ and $m=|E(G)|$. Then

$$i_2(G) \ge \left\lceil (m+1)/n \right\rceil$$

and this inequality is sharp.∎

How good is this estimate? Let H be a graph with large interval number and let $G=H+pK_1$, where p is very large. Then for G the ratio $(m+1)/n$ can be made arbitrarily close to 0. One is then led to ask: Suppose G were connected, or had large connectivity κ, is the above estimate more useful? Unfortunately, the above formula cannot be guaranteed to provide a close estimate:

Let $X_1,X_2,\ldots,X_s$ be disjoint finite sets with $|X_1|=|X_2|=p$ and $|X_i|=k$ for $3\le i\le s$. Let G be the graph defined by:

$V(G) = X_1\cup\ldots\cup X_s$

$E(G) = \{vw\colon\ v\in X_i \text{ and } w\in X_{i+1},\ \text{for some } i,\ 1\le i\le s-1\}.$

We assume $s\gg p\gg k$. Observe that G is triangle-free, $\kappa(G)=k$ and $i(G) = i_2(G) \ge i(K_{p,p}) \ge \frac{1}{2}p$. However, $n = |V(G)| = 2p+(s-2)k$ and $m = |E(G)| = p^2+(s-1)k^2$. Thus $(m+1)/n = [p^2+(s-2)k^2+1]/[2p+(s-2)k] < k+1$ for s sufficiently large.

A much more accurate estimate for $i_2(G)$ involves the arboricity of a graph which we define now:

2.2.5 Definition. Let $G=(V,E)$ be a graph. The **arboricity** of G, denoted $T(G)$, is the least positive integer a so that $E=E_1\cup E_2\cup\ldots\cup E_a$ and (V,E_i) is a tree for $i=1,2,\ldots,a$.

2.2.6 Theorem [67]. For any graph G

$i_2^+(G) \le T(G)+1.$

Proof. Let $a=T(G)$ and let $E=E_1\cup E_2\cup\ldots\cup E_a$ be a covering of G's edges by trees. Let $v_0\in V(G)$. For each tree $T_i=(V,E_i)$ let all edges be directed toward v_0. Thus each edge has out-degree at most 1 in T_i, $i=1,\ldots,a$. Assign to each vertex in G an interval so that these "primary" intervals are disjoint. Notice that the intervals have non-empty interior–they are dislayed. For each i, a vertex v has a unique neighbor w in T_i with

$v \to w$. Place an interval for v in the displayed portion of w's primary interval. (Notice that the representation is still displayed.) Continuing this procedure for each vertex and each $i=1,2,\ldots,a$ we form a displayed $(a+1)$-interval representation with depth-2 for G. See figure 2.2.∎

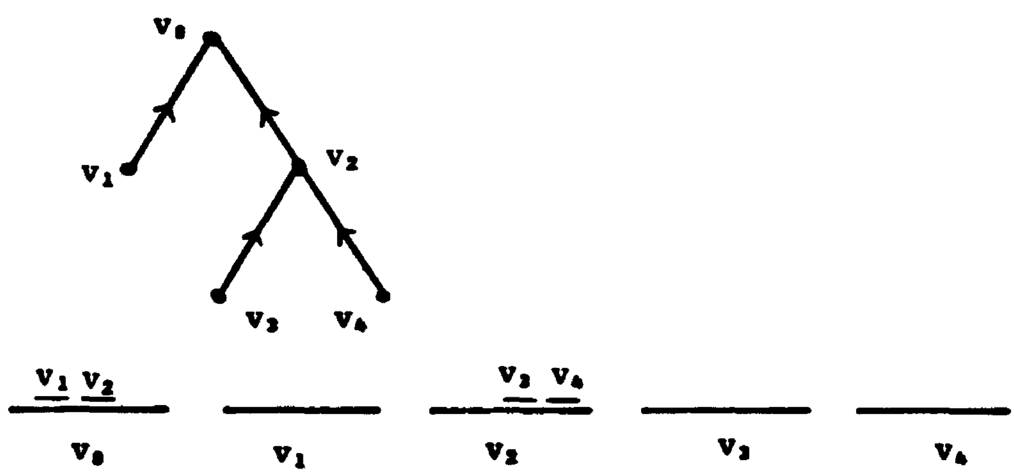


Figure 2.2

2.2.7 Theorem [11,54,55]. Let G be a graph.
$T(G) = \max\{ \lceil |E(H)|/(|V(H)|-1) \rceil : H \le G\}$.∎

2.2.8 Theorem. Let G be a graph, then
$i_2(G) \ge T(G)-1$.

Proof. By 2.2.7 there is an $H \le G$ with $n=|V(H)|$, $m=|E(H)|$ and $T(G)=\lceil m/(n-1) \rceil$. Also $i_2(G) \ge i_2(H) \ge \lceil (m+1)/n \rceil$ by 2.2.4. The result follows once we show $\lceil (m+1)/n \rceil \ge \lceil m/(n-1) \rceil -1$. Suppose

$$\lceil (m+1)/n \rceil < \lceil m/(n-1) \rceil -1 = \lceil [m/(n-1)] -1 \rceil.$$

Thus

$$\lceil [(m+1)(n-1)]/[n(n-1)] \rceil < \lceil [(m-n+1)(n)]/[n(n-1)] \rceil, \text{ or}$$
$$(m+1)(n-1) < (m-n+1)(n), \text{ or}$$

$mn+n-m-1 < mn-n^2+n$, or

$-m-1 < -n^2$, or

$m > n^2-1$.

But $m \le \binom{n}{2} = \frac{1}{2}n(n-1)$ which would imply

$\frac{1}{2}(n)(n-1) > (n-1)(n+1)$, or

$\frac{1}{2}n > n+1$

which is impossible.∎

2.2.9 Corollary. For any graph G, $|i_2(G) - T(G)| \le 1$.

Proof. Immediate from 2.2.6 and 2.2.8 since $T(G^+)=T(G)$.∎

2.2.10 Remark. The above corollary is important in two respects. First it establishes the very close relationship between i_2 and T. More importantly, however, is that the arboricity is readily computable—Kameda [42] and Edmonds [11] give polynomial time algorithms for its determination. Since $i_2=i$ for triangle-free graphs and $i_2 \ge i$ always, this result is very useful.

The inequalities implicit in 2.2.9 are sharp and $i_2(G)-T(G)$ can take on each of the values -1,0,+1 as the following examples show:

(1) Let G be a caterpillar. Then $i_2(G) = T(G) = 1$.

(2) Let $G=K_{3,5}$. Then $i_2(G)=2$ and $T(G)=3$.

(3) Let $G=(K_{2,9})^+$. By 5.2.7 we will see that $i_2(G)=3$. It is readily verified that $T(G)=2$.

For triangle-free graphs G we know $i_2(G)=i(G)$ and so 2.2.3, 2.2.4 and 2.2.9 provide excellent information on the interval number of such graphs. On the other end of the graph theory spectrum from triangle-free graphs are the chordal graphs. Since chordal graphs without asteroidal triples are the interval graphs (see §1.1.11) one might suspect that chordal graphs would have bounded interval number. This,

however, is false. The key ideas in the following argument are due to Trotter:

2.2.11 Theorem [67]. For every positive integer t there exists a chordal graph G with $i(G)>t$.

Proof. Let t be a fixed positive integer and let $n=(t+1)(2t+1)+1$. Define a graph G with vertex set $X \cup Y$ where:

$X = \{x_1,\ldots,x_n\}$ and

$Y = \{y_I: \ I \subset \{1,\ldots,n\}$ and $|I|=t+1\}$.

Furthermore, we let X be an independent set and Y be a clique and $x_i \sim y_I$ if and only if $i \in I$. Observe that G is chordal (indeed, G is a split graph). Suppose $i(G) \le t$ and fix a t-interval representation f for G.

In this representation the (at most) nt intervals for the vertices in X are pairwise disjoint and therefore appear in some order on the real line. Define a graph H with $V(H)=X$ and $x_i \sim x_j$ in H if and only if some interval for x_i is immediately adjacent to some interval for x_j in f. Observe that $|V(H)|=n$ and $\Delta(H) \le 2t$. We claim that $\alpha(H)>t$:

Construct an independent set of cardinality t+1 in H as follows: Choose $x_{i_1} \in X_1 = X$. It has at most 2t neighbors in H. Delete x_{i_1} and its neighbors from X_1 and choose:

$x_{i_2} \in X_2 = X_1 - \{x_{i_1}\} - \mathrm{adj}(x_{i_1})$.

Continue in this fashion choosing

$x_{i_p} \in X_p = X_{p-1} - \{x_{i_{p-1}}\} - \mathrm{adj}(x_{i_{p-1}})$.

Observe that $\{x_{i_1}, x_{i_2}, \ldots\}$ forms an independent set in H. Since

$|X_p| \ge |X_{p-1}| - (2t+1) \ge n - p(2t+1)$,

we see that $|X_{t+1}| \ge n-(t+1)(2t+1) \ge 1$. Thus $X_{t+1} \ne \emptyset$, and so we can construct an independent set of vertices of cardinality t+1.

Thus we can find a $J \subset \{1,\ldots,n\}$ so that $|J|=t+1$ and $\{x_j: j \in J\}$ is

independent in H. Suppose $f(y_J)=I_1\cup\ldots\cup I_t$. We check that each I_k can intersect at most one of $f(x_j)$ with $j\in J$. This is now impossible since y_J is adjacent to t+1 vertices x_j, $j\in J$.∎

2.2.12 Remark. For both chordal graphs and triangle-free graphs one cannot bound the interval number. The usual examples in the latter case are the complete bipartite graphs. It is therefore logical to ask what bounds can be placed on the interval number of a graph with given girth (length of shortest cycle). For graphs with girth 3 and 4 we know that there is no bound on the interval number. This remains true for graphs with arbitrary girth:

2.2.13 Theorem. Given positive integers $g\geq3$ and t there exists a graph G with girth g and $i(G)\geq t$.

Proof. It is known that there exist k-regular graphs with arbitrary girth [14]. Such a graph has interval number $\lceil\frac{1}{2}(k+1)\rceil$.∎

Next we relate the interval number to the intersection number which we define presently.

2.2.14 Definition. Let G be a graph with $V(G)=\{v_1,\ldots,v_n\}$ and $f:V(G)\rightarrow S$ be an intersection representation. Denote by $\cup f$ the union $f(v_1)\cup\ldots\cup f(v_n)$. The **intersection number** of G, denoted I(G), is the smallest integer t for which there exists a representation f for G with $|\cup f|=t$.

2.2.15 Theorem [13]. The intersection number of a graph G equals the size of a minimal edge cover by cliques, i.e. $E(G)=E_1\cup\ldots\cup E_{I(G)}$ and $(V(G),E_j)$ is a clique plus isolated vertices and no smaller cover exists.∎

2.2.16 For any graph G we have $i(G) \le \lceil \frac{1}{2}(1+I(G)) \rceil$.

Proof. Let f_0 be a representation of G with $f_0(v) \subset \{1,\dots,I(G)\}$ for all $v \in V(G)$. We define a multiple interval representation f for G putting f(v) to be a minimal collection of intervals containing $f_0(v)$ and no other integers. For example, if $f_0(v)=\{2,3,4,6,8,9\}$ then put $f(v)=[2,4] \cup [5.9,6.1] \cup [8,9]$. Clearly f is a multiple interval representation for G and in the worst case we have, for example, $f_0(v)=\{1,3,5,\dots\}$. Then f(v) contains $\lceil \frac{1}{2}(1+I(G)) \rceil$ intervals.∎

2.2.17 Remark. We have seen that there are several graph parameters which convey information about the interval number (such as $|V|$, $|E|$, T, Υ and Δ) and several which do not (such as κ, α, girth and χ). We consider the effect of various graph operations on the interval number next.

2.2.18 Theorem. There is no relationship between i(G) and $i(\bar{G})$.

Proof. If $G=2K_n$ then $i(G)=1$, but $i(\bar{G})=i(K_{n,n}) \ge \frac{1}{2}n$. If $G=K_n$ then $i(G)=i(\bar{G})=1$.∎

2.2.19 Theorem. For any graph G its subdivision satisfies $i(\beta G) \le 2$.

Proof. Obvious.∎

2.2.20 Theorem. For any graph G its line graph satisfies $i(L(G)) \le 2$.

Proof. Let $V(G)=\{v_1,\dots,v_n\}$ and define $f:V(G) \to 2^{B^1}$ by:

$$f(v_i v_j) = [i-\tfrac{1}{4},i+\tfrac{1}{4}] \cup [j-\tfrac{1}{4},j+\tfrac{1}{4}].$$

Clearly f is a 2-interval representation for G.∎

2.2.21 Theorem. For any integer t there exist interval graphs G and H so that $i(G \vee H)=t$ (see Definition 0.2.6 for join).

Proof. $nK_1 \vee mK_1 = K_{n,m}$. ▮

2.2.22 Definition. Let G_1 and G_2 be graphs. Their **product**, denoted $G_1 \times G_2$, is given by:

$V(G_1 \times G_2) = V(G_1) \times V(G_2)$, and

$(x_1,x_2) \sim (y_1,y_2)$ if and only if either $x_1 \sim x_2$ and $y_1 = y_2$, or else $x_1 = x_2$ and $y_1 \sim y_2$.

One checks that this product is associative. For a positive integer n one defines G^n by:

$G^1 = G$, and

$G^n = G^{n-1} \times G$, for $n>1$.

2.2.23 Theorem. For all graphs G_1 and G_2 we have $i(G_1 \times G_2) \le i(G_1)+i(G_2)$.

Proof. Let $i(G_j)=k_j$, $j=1,2$. One can represent the edges of the form $(x,z)(y,z)$ using k_1 intervals per vertex since this subgraph consists of $|V(G_2)|$ disjoint copies of G_1. Similarly we can represent the edges of the form $(x,y)(x,z)$ with k_2 intervals per vertex and the result follows. ▮

2.2.24 Remark. The k-cube is defined by $(K_2)^k$ and the above result implies that its interval number is at most k. However, it is known [30] that its interval number is $\lceil \frac{1}{2}(k+1) \rceil$. Nevertheless, the above result is sharp in the following strong sense:

2.2.25 Theorem. For every pair of positive integers k_1, k_2 there exists graphs with $i(G_j)=k_j$, $j=1,2$ and $i(G_1 \times G_2) = k_1+k_2$

Proof. Let P_n denote the path graph with n vertices and $P_n^k=(P_n)^k$. Let $v_n(k)=|V(P_n^k)|$ and $e_n(k)=|E(P_n^k)|$. We determine the functions v_a and e_n:

Since $P_n^k=P_n^{k-1}\times P_n$, one has:

$$v_n(k) = nv_n(k-1), \text{ and}$$
$$e_n(k) = ne_n(k-1) + (n-1)v_n(k-1).$$

Given that $v_n(1)=n$ and $e_n(1)=n-1$ one checks (e.g. by induction) that the solution to the above recurrence relation is:

$$v_n(k) = n^k, \text{ and}$$
$$e_n(k) = k(n-1)n^{k-1}.$$

We consider the case n>k and claim $i(P_n^k)=k$.

Since P_n^k is triangle-free we have by 2.2.4:

$$\begin{aligned} i(P_n^k) &= i_2(P_n^k) \geq \\ &\geq \lceil [e_n(k)+1]/v_n(k)\rceil = \\ &= \lceil [k(n-1)n^{k-1}+1]/n^k\rceil \geq \\ &\geq \lceil k(n-1)/n\rceil = \lceil k - k/n\rceil = k. \end{aligned}$$

Thus $i(P_n^k)\geq k$, but by 2.2.23, $i(P_n^k)\leq k$, proving the claim.

Now given k_1,k_2, let $n=k_1+k_2+1$, $G_1=P_n^{k_1}$ and $G_2=P_n^{k_2}$. Thus $i(G_j)=k_j$ and $i(G_1\times G_2)=k_1+k_2$.∎

We conclude with a tabluation of interval number formulae and relations (some will be proved in later sections).

Interval Number Formulae

$i(G):=\inf\{t\colon G\in\Omega(t\{[a,b]\colon a,b\in\mathbf{R},\ a<b\})\}$

$i(G^{+})=i^{+}(G)$

$i(G)\le i^{+}(G)\le i(G)+1$

$i_2(G)\ge i_3(G)\ge\ldots\ge i(G)$

$i_2(G)\le(\omega(G)-1)i(G)$

$i(K_n)=1$

$i_2(K_n)=\lceil \frac{1}{2}n\rceil$

$i_r(K_n)=\frac{1}{2}[n/(r-1)]+o(n)$

$i(K_{n,m})=\lceil (nm+1)/(n+m)\rceil$

$i(T)\le 2$, T a tree

$i(G)\le\lceil \frac{1}{2}(|V(G)|+1)\rceil$

$i(G)\le\sqrt{|E(G)|}$

$i_2(G)\le\lceil \frac{1}{2}(\Delta(G)+1)\rceil$

$i_2(G)\ge\lceil (|E(G)|+1)/|V(G)|\rceil$

$|i_2(G)-T(G)|\le 1$

$i(G)\le\lceil \frac{1}{2}(1+I(G))\rceil$

$i(G)\le 3$, G planar

$i(G)\le\sqrt{(3\gamma(G))}$, G non-planar $(\gamma>0)$

$i(G+H)=\max\{i(G),i(H)\}$

$i(G\times H)\le i(G)+i(H)$, $i(P_n^k)=k$ for $k<n$

$i(\beta G)\le 2$

$i(L(G))\le 2$

3. INTERVAL NUMBER VARIANTS

In this chapter we discuss two problems both of which concern the behavior of parameters closely related to the interval number. The first deals with calculating the depth-r interval number for complete graphs. The second concerns the inefficiency of "non-redundant" multiple interval representations.

3.1 Depth-r Interval Number of Complete Graphs

The interval number of K_n is clearly 1; we simply assign the same interval, say [0,1], to all the vertices. However, if we require a multiple interval representation of K_n to have depth less than n, the Helly property (§3.2.1) implies that one interval per vertex will not suffice. We attack the problem of computing $i_r(K_n)$. In case $r=2$ we give exact results. For $r>2$ we give upper and lower bounds that are close together.

3.1.1 Theorem. $i_2(K_n) = \lceil \frac{1}{2}n \rceil$.

Proof. It is well known that the edges of K_n can be covered by $\lceil \frac{1}{2}n \rceil$ paths. Since paths are triangle-free interval graphs we know that $i_2(K_n) \le \lceil \frac{1}{2}n \rceil$.

Suppose we have a depth-2 t-interval representation for K_n. There are a total of nt intervals. The maximum number of edges these intervals can form can be enumerated by counting left endpoints; every intersection of two intervals contains exactly one left endpoint. There are nt left endpoints and the very leftmost cannot be contained in another interval. Hence we can represent at most nt-1 edges. Thus,

$$nt - 1 \ge \binom{n}{2},$$
$$nt \ge \tfrac{1}{2}(n)(n-1) + 1, \text{ and hence}$$

$$t \geq \tfrac{1}{2}(n-1) + 1/n.$$

Since t is an integer we have:

$$t \geq \left\lceil \tfrac{1}{2}(n-1) + 1/n \right\rceil = \left\lceil \tfrac{1}{2}n \right\rceil. \blacksquare$$

Thus we have an exact value for $i_2(K_n)$. We now consider greater depth. First a lower bound:

3.1.2 Theorem. $i_r(K_n) \geq \left\lceil \tfrac{1}{2}\left(\frac{n-1}{r-1} + \frac{r}{n}\right) \right\rceil$.

Proof. Choose a t-interval depth-r representation for K_n. We enumerate the maximum number of edges we can represent. Each left endpoint of the nt intervals is contained in at most r-1 intervals without violating depth-r. However, the very leftmost left endpoint is in 0 other intervals, the next left endpoint is in at most 1 other interval, the next in at most 2, etc. Thus the first r-1 intervals (ordered by left endpoints) make (r-1), (r-2),..., 2, 1 fewer intersections than the maximum r-1 possible; this gives a total deficiency of $\binom{r}{2}$. Hence,

$$|E| = \binom{n}{2} \leq nt(r-1) - \binom{r}{2}.$$

Thus, $nt(r-1) \geq \binom{n}{2} + \binom{r}{2}$, or

$$t \geq \tfrac{1}{2}\left(\frac{n-1}{r-1} + \frac{r}{n}\right)$$

and since t is an integer,

$$t \geq \left\lceil \tfrac{1}{2}\left(\frac{n-1}{r-1} + \frac{r}{n}\right) \right\rceil. \blacksquare$$

In case r=2 this expression reduces to $\left\lceil \tfrac{1}{2}n \right\rceil$ in agreement with 3.1.1.

The first inequality in the above proof was independently found in [50]. The approach there was slightly different. It was shown that for a graph G with n vertices, m edges and maximum clique of size ω that:

$$i(G) \leq \left\lceil \frac{m+\frac{1}{2}\omega(\omega-1)}{n(\omega-1)} \right\rceil.$$

Next we obtain a recursive upper bound:

3.1.3 Theorem. $i_r(K_n) \le i_r(K_{\lceil n/r \rceil}) + \lceil \frac{1}{2}(\lceil \frac{n}{r} \rceil + 1) \rceil$.

Proof. We first partition the vertices of K_n into r roughly equal size sets, each with at most $\lceil n/r \rceil$ vertices. The edges joining vertices within each one of these sets form an $\lceil n/r \rceil$-clique and we form a depth-r representation for these cliques with at most $i_r(K_{\lceil n/r \rceil})$ intervals per vertex. The edges we have yet to represent join vertices in different parts, i.e., they form a complete multipartite graph G with r parts and each part of size at most $\lceil n/r \rceil$. The largest possible clique in G has r vertices and therefore $i_r(G)=i(G)$. By Theorem 2.1.12 we can represent these edges using $\lceil \frac{1}{2}(\lceil n/r \rceil + 1) \rceil$ intervals per vertex. Since every edge of K_n is either within one of the r sets or between a pair, the result follows.∎

In spite of the repeated use of the ceiling function we can obtain an upper bound in closed form. We first derive a less precise recursive upper bound:

3.1.4 Theorem. $i_r(K_n) \le i_r(K_{\lfloor n/r \rfloor}) + \lceil \frac{1}{2}(\lfloor n/r \rfloor + 1) \rceil + 1$.

Proof. We partition the vertices of K_n into r parts with exactly $\lfloor n/r \rfloor$ vertices and one "remainder" part with the remaining $<r$ vertices. As in the previous proof we represent the edges inside the pieces of size $\lfloor n/r \rfloor$ with $i_r(K_{\lfloor n/r \rfloor})$ intervals per vertex and between those pieces with $\lceil \frac{1}{2}(\lfloor n/r \rfloor + 1) \rceil$ intervals per vertex and not violate the depth-r restriction. We still need to represent the edges inside the "remainder", and between the "remainder" and the rest of the graph. This can be done with one interval per vertex as shown in Figure 3.1.∎

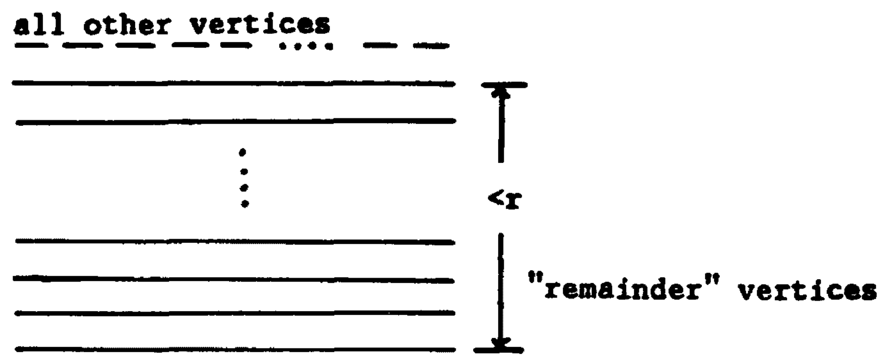


Figure 3.1

3.1.5 Remark. the estimate in §3.1.3 is, in general, better than that of §3.1.4. By §3.1.3 we have $i_3(K_8)\le 3$ but by §3.1.4 we have $i_3(K_8)\le 4$. The advantage of the latter result is that it is more tractable in deriving a closed-form upper bound.

3.1.6 Theorem. $i_r(K_n) \le \frac{n}{2(r-1)} + 2\lceil \log_r n \rceil + 1.$

Proof. We observe that $\lceil a/b \rceil \le (a+b-1)/b$ and that $\lfloor a/b \rfloor \le a/b$ for all positive integers a and b. Fix r. Let $f(x) = i_r(K_{\lfloor x \rfloor})$. We therefore have,

$$\begin{aligned} f(n) &\le f(n/r) + \lceil \tfrac{1}{2}((n/r)+1) \rceil + 1 \le \\ &\le f(n/r) + \tfrac{1}{2}(n/r + 1 + 1) + 1 = \\ &= f(n/r) + n/2r + 2. \end{aligned}$$

We expand this recurrence relation to $\lceil \log_r n \rceil$ terms:

$$f(n) \le (n/2r + n/2r^2 + n/2r^3 + \dots) + (2 + 2 + \dots + 2) + f(\varepsilon)$$

with $\varepsilon \le 1$. Thus,

$$\begin{aligned} f(n) &\le \left(\frac{n}{2r}\right)\left(\frac{1}{1-(1/r)}\right) + 2\lceil \log_r n \rceil + 1 = \\ &= \frac{n}{2(r-1)} + 2\lceil \log_r n \rceil + 1. \blacksquare \end{aligned}$$

This can be improved, of course, if n is a power of r:

3.1.7 Theorem. If n is a power of r then $i_r(K_n) \le \frac{1}{2}(\frac{n-1}{r-1}) + c(\log_r n)$ with c=1 if n is even and c=½ if n is odd.

Proof. Let $f(k) = i_r(K_n)$ for $n=r^k$. By 3.1.3,

$$f(k) \le f(k-1) + \left\lceil \tfrac{1}{2}(r^{k-1}+1) \right\rceil \le$$
$$\le f(k-1) + \tfrac{1}{2}r^{k-1} + c$$

with c equal to 1 or ½ as r is even or odd. Expanding we obtain:

$$f(k) \le (\tfrac{1}{2}r^{k-1}+c) + (\tfrac{1}{2}r^{k-2}+c) + \ldots + (\tfrac{1}{2}+c) =$$
$$= \tfrac{1}{2}(\tfrac{n-1}{r-1}) + ck. \blacksquare$$

For n>>r the upper and lower bound are assymptotically the same:

3.1.8 Corollary. For r fixed we have $i_r(K_n) = \frac{1}{2}(\frac{n}{r-1}) + o(n)$.

Proof. Immediate from 3.1.2 and 3.1.6.$\blacksquare$

3.1.9 Remark. None of the formulae in the upper and lower bounds give a correct expression for $i_r(K_n)$. A non-trivial example is the calculation of $i_3(K_8)$. By §3.1.2 $i_3(K_8) \ge \lceil \frac{1}{2}(7/2 + 3/8) \rceil = \lceil 31/16 \rceil = 2$. The fraction 31/16 is already very close to 2 and as a result there is very little "flexibility" in constructing a depth-3 2-interval representation

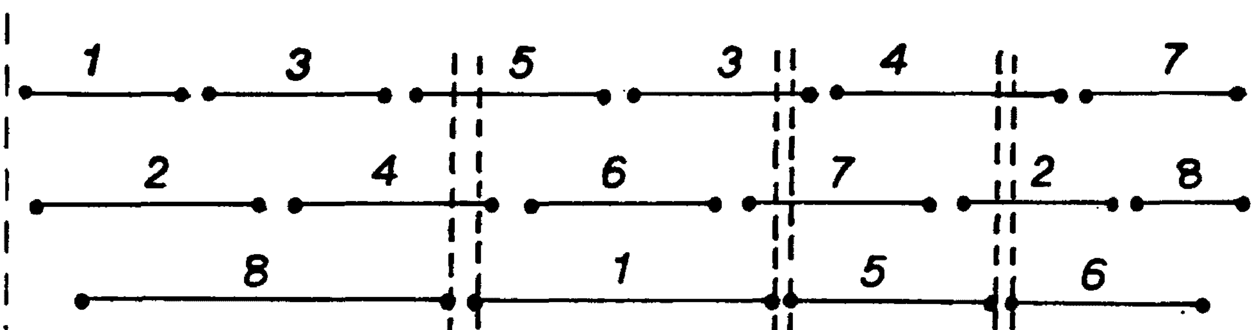


Figure 3.2

of K_8. Such a representation appears in Figure 3.2. As we saw,

however, in §3.1.5 the best upper bound estimate for $i_3(K_8)$ is 3. This might lead one to conjecture that the lower bound formula always gives $i_r(K_n)$. This, however, is false as we shall see in section 3.2.

We have a good idea of the behavior of $i_r(K_n)$ when $n>>r$. We conclude this section with a simple result for when r is "large".

3.1.10 Theorem. For $n>r>n/2$ we have $i_r(K_n) = 2$.

Proof. We construct a 2-interval representation of K_n with depth $r=\lfloor \frac{1}{2}n \rfloor+1$. If $r'<r$, clearly $r'\leq\frac{1}{2}n$ and the theorem will follow. Number the vertices of K_n from 1 to n. Assign to vertices 1 through $\lfloor \frac{1}{2}n \rfloor$ the same interval, say [0,1]. To the remaining $\lceil \frac{1}{2}n \rceil$ vertices assign non-overlapping intervals inside [0,1] and also, to those last $\lceil \frac{1}{2}n \rceil$, the

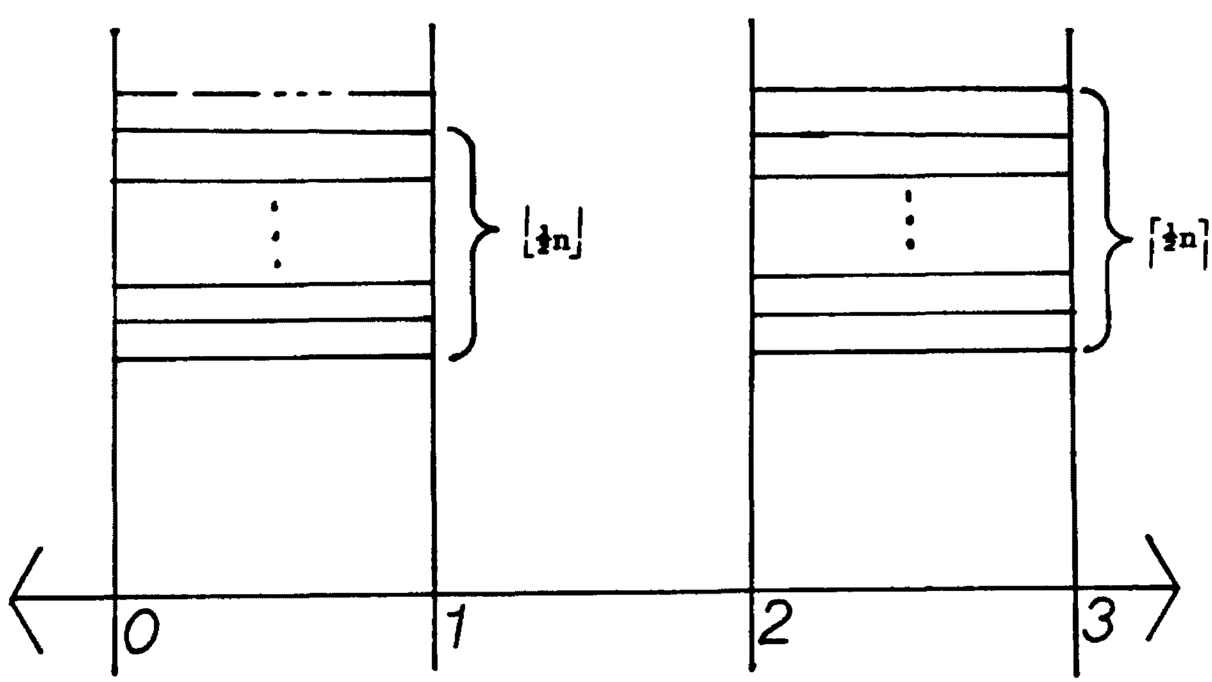


Figure 3.3

interval [2,3]. See Figure 3.3. Observe that this is a 2-interval representation of K_n with depth $\lfloor \frac{1}{2}n \rfloor + 1$.■

3.2 Variations on Helly's Theorem

Relatively little of interest can be said about an interval representation of a complete graph. The most interesting is Helly's theorem:

3.2.1 Theorem [37]. Given a finite family of real intervals which intersect pairwise, there exists a real number which is contained in all of these intervals.■

This can be rephrased as follows:

3.2.2 Corollary. Every 1-interval representation of K_n has depth-n.■

The results of the previous section can be used to extend this observation to multiple interval representations of K_n.

3.2.3 Theorem. Every t-interval representation of K_n has depth exceeding n/2t.

Proof. By 3.1.2 we compute:

$$t \geq i_r(K_n) \geq \left\lceil \tfrac{1}{2}\left(\frac{n-1}{r-1} + \frac{r}{n}\right) \right\rceil > \tfrac{1}{2}\left(\frac{n-1}{r-1}\right).$$

Hence, $(r-1) > (n-1)/2t$, or $r > (n-1)/2t + 1 > n/2t$.■

3.2.4 Remark. Theorem 3.2.3 can be viewed as a Ramsey-type theorem: for fixed t, a t-interval representation of K_n must contain a "large" accumulation of intervals at one point on the real line. The relative size of this accumulation is independent of n. A t-interval representation of K_n cannot be randomly scattered but must "pile" up in

locations. A similar result has been proved in [31]. Our notation is an adaptation of theirs:

3.2.5 Definition. Let $f:V(G) \to S$ be a intersection representation. A **transversal** for f is a set of points P, so that $f(v) \cap P \neq \emptyset$ for all $v \in V(G)$. Denote by $\tau(f)$ the minimum cardinality of a transversal for f.

3.2.6 Definition. Let t be a positive integer. Denote by $L^*(t)$ the least positive integer so that if f is any t-interval representation for a complete graph, then $\tau(f) \leq L^*(t)$. In symbols,

$$L^*(t) = \sup\{\tau(f): f:K_n \to tB^1 \text{ is a representation}\}$$

The surprise is:

3.2.7 Theorem [31]. $L^*(t)$ is finite for every positive integer t.∎

3.2.8 Remark. For $t=1$ we have $L^*(1)=1$, which is Helly's theorem. [31] shows that $L^*(2)=3$. This implies that for every 2-interval representation of K_n we have three points such that each "interval pair" meets at least one of these points. Hence, the depth at one of these points must be at least $n/3$. By §3.2.3 we only knew that the depth of this representation must exceed $n/4$.

That $L^*(2)=3$ implies that the lower bound in Theorem 3.1.2 is not the "true" formula for $i_r(K_n)$. For example, consider $i_6(K_{19})$. By 3.1.2 we have $i_6(K_{19}) \leq \lceil \frac{1}{2}(18/5 + 6/19) \rceil = \lceil 186/97 \rceil = 2$. Since $19/3>6$, in any 2-interval representation of K_{19}, the depth must be at least 7. Thus $i_6(K_{19})>2$ and the formula in §3.1.2 does not give $i_r(K_n)$.

[31] proves that $L^*(t)$ is finite by giving an upper bound. Unfortunately, this bound grows superexponentially and it seems doubtful that L^* should grow that fast. Indeed, based on Theorem 3.2.3 and that $L^*(1)=1$ and $L^*(2)=3$, we venture the following:

3.2.9 Conjecture. $L^*(t)\le 2t$. Perhaps even $L^*(t)=2t-1$.

Unfortunately, the computation of L^* values seems quite difficult.

3.3 Non-redundant Interval Representations

When two real intervals intersect, the intersection is an interval. However, if two double intervals meet, the intersection may consist of 1, 2 or 3 intervals. In a multiple interval representation of a graph it is only necessary for one interval assigned to a vertex to meet one interval of an adjacent vertex. Any additional intersections are permitted, but are "superfluous" or "redundant". If we disallow "redundancies" in multiple interval representations we arrive at the concept of non-redundant multiple interval representation. Every 1-interval representation is necessarily non-redundant. The same statement for multiple interval representations is obviously false. We ask instead: is there any "harm" in requiring multiple interval representations to be non-redundant? More precisely, if G has a t-interval representation, does it also possess a non-redundant t-interval representation? We shall see that the answer is "no" and that the "harm" can be devastating.

3.3.1 Definition. A t-interval representation of graph $f:V(G)\to t\mathbf{B}^1$ is **non-redundant** if $f(v)\cap f(w)$ is either empty or an interval for all vertices $v\ne w$.

3.3.2 Definition. Let G be a graph. The **non-redundant interval number** of G, denoted NRI(G), is the least positive integer t for which G has a non-redundant t-interval representation.

Clearly $NRI(G)\ge i(G)$ since a non-redundant t-interval representation is, by definition, a t-interval representation. However,

it is not necessary that $i(G) \geq NRI(G)$, as we now see.

3.3.3 Definition. Let G be a graph. We say that G is **t-tight** if every t-interval representation of G, $f:V(G) \rightarrow tB^1$, has the property that the union of all the sets $f(v)$, $\cup f$, is a single interval.

3.3.4 Lemma [79]. $K_{3,5}$ is 2-tight.∎

This can be verified by noting that if there is any gap in a 2-interval representation for $K_{3,5}$, then at most 14 edges can be represented.

Also, it can be easily verified that any vertex of $K_{3,5}$ can appear as the "first" interval (ordered by left endpoint) in a 2-interval representation for $K_{3,5}$.

3.3.5 Theorem. There exists a graph G for which $i(G) < NRI(G)$.

Proof. We explicitly construct such a graph. Let G be the graph consisting of 4 disjoint copies of $K_{3,5}$, each with a distinguished vertex, plus two additional vertices x and y which are adjacent to each other and to each of the four distinguished vertices. See Figure 3.4. Recalling the last sentence of 3.3.4 it is now easy to see that $i(G)=2$. Such a representation appears in Figure 3.5. Since the $K_{3,5}$'s are 2-tight they must appear as unbroken intervals in G's interval representation. In order for x and y to be adjacent to the four distinguished vertices their intervals must span the two gaps between the first and last pairs of $K_{3,5}$'s as they sit on the real line. Thus the edge xy must be represented "twice" in every 2-interval representation of G. Thus G has no non-redundant 2-interval representation and hence $NRI(G) > i(G) = 2$.∎

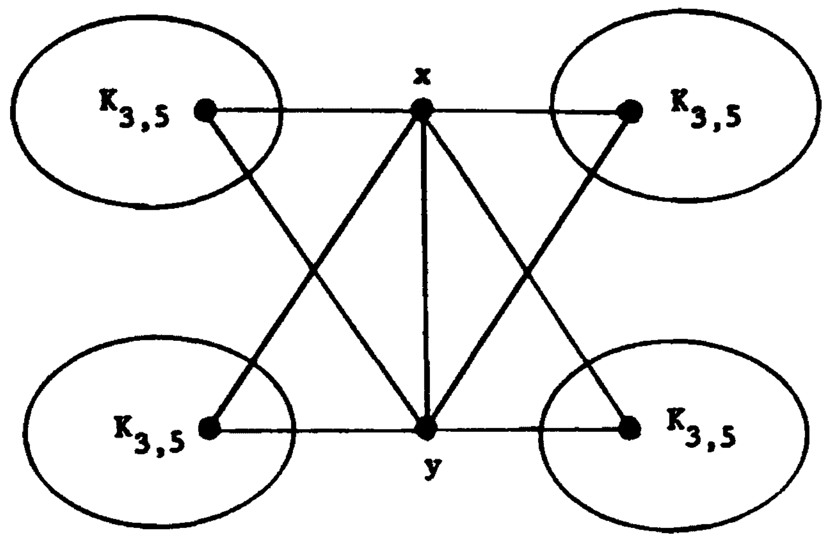


Figure 3.4

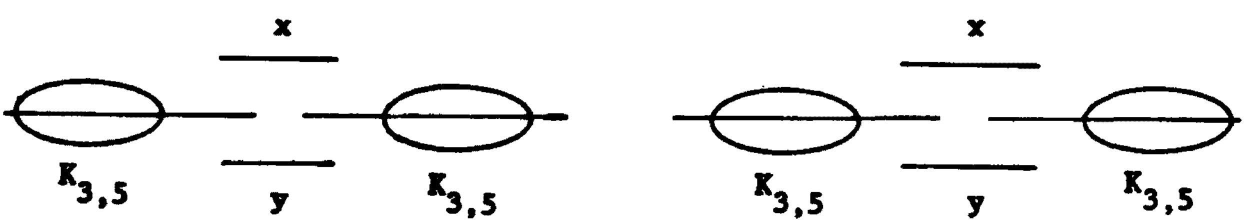


Figure 3.5

3.3.6 Remark. It is trivial to show that NRI(G)=3 for the graph in the preceding proof. It can easily be shown that $K_{2t-1,2t+1}$ is t-tight and we can construct graphs G with i(G)=t and NRI(G)=t+1 for every integer t≥2 by analogy with our construction in §3.3.5.

The 2-tightness of $K_{3,5}$ was critical in constructing the previous example. Clearly no t-tight graph is also (t+1)-tight. Hence the structure, which forced an edge such as xy in our example to be

represented "more than once" in a t-interval representation, evaporates in a (t+1)-interval representation. One is led to suspect that the non-redundant interval number might satisfy: $NRI(G) \le i(G)+1$. Indeed, nothing could be further from the truth:

3.3.7 Theorem. For every positive integer t there exists a graph G with $i(G)=2$ and $NRI(G)>t$.

The proof relies on a repeated use of the following "probabilistic" pigeon-hole result.

3.3.8 Lemma. Let $0<\varepsilon<1$ and $\delta=\frac{1}{2}\varepsilon$. Let $C(1),C(2),\dots,C(p)$ be disjoint finite sets each of cardinality q, and let $C(*)$ denote their union. Suppose $S\subset C(*)$ and $|S|\ge\varepsilon|C(*)|=\varepsilon pq$. Then the number of indices k for which

(1) $\quad |S\cap C(k)|\ge\delta|C(k)|=\delta q$

is at least δp, i.e.

$$|\{k:\ |S\cap C(k)|\ge\delta|C(k)|\}| \ge \delta p.$$

Proof. Suppose fewer than δp of the $C(k)$ satisfy (1). That means up to $\delta p-1$ of the $C(k)$ can have "a lot" (but at most $q=|C(k)|$) of their elements in S while the remaining $p-\delta p+1$ $C(k)$'s can have at most $\delta q-1$ elements in S. Thus,

$$\begin{aligned}|S| &\le (\delta p-1)q + (p-\delta p+1)(\delta q-1) = \\ &= \delta pq - q + p\delta q - \delta^2 pq + \delta q - p + \delta p - 1 = \\ &= 2\delta pq - (1-\delta)p - (1-\delta)q - \delta^2 pq - 1 < \\ &< 2\delta pq = \varepsilon pq.\end{aligned}$$

But $|S|\ge\varepsilon pq$ by hypothesis and the lemma follows.∎

3.3.9 Proof of 3.3.7. We define graphs $G(n,m,q)$, where n,m,q are positive integers, as follows: The vertices of $G(n,m,q)$ are triples of integers (i,j,k) with $1\le i\le n$, $1\le j\le m$ and $1\le k\le q$. In symbols,

$$V(G(n,m,q)) = \{1,\dots,n\} \times \{1,\dots,m\} \times \{1,\dots,q\}.$$

We put $(i,j,k)\sim(i',j',k')$ if and only if $i=i'$ or $j=j'$. (Notice that adjacency does not involve the third coordinate. Thus $(i,j,k)\equiv(i,j,k')$ for all i,j,k,k'.) One readily verifies that $\rho G(n,m,q)=K_n\times K_m$. Thus $i(G(n,m,q)) = i(\rho G(n,m,q)) = i(K_n\times K_m) \le i(K_n)+i(K_m) = 2$. In case $n,m>1$, then $C_4\le G(n,m,q)$ and so $i(G(n,m,q))=2$.

Let t be a positive integer. We show that for suitable n,m_0,q_0 we have $NRI(G(n,m_0,q_0))>t$. Here suitable will entail $t\ll n\ll m_0\ll q_0$. In particular, we may choose

$$n = 12t$$
$$m_0 = 2(4t)^n$$
$$q_0 = 2(4t)^{n+1}$$

Let $G_0=G(n,m_0,q_0)$. Observe that for all i,j the set of vertices $C_0(i,j)=\{(i,j,k):1\le k\le q_0\}$ is a clique containing q_0 vertices. Also $C_0(i,*)=C_0(i,1)\cup\dots\cup C_0(i,m_0)$ is a clique with m_0q_0 vertices and $C_0(*,j)=C_0(1,j)\cup\dots\cup C_0(n,j)$ is a clique with nq_0 vertices.

Suppose $NRI(G_0)\le t$. Fix a t-interval non-redundant representation f for G_0. Let $\varepsilon=1/(2t)$ and $\delta=\frac{1}{2}\varepsilon$. Consider the clique $C_0(1,*)$. By 3.2.3 its representation in f has depth at least $\varepsilon|C_0(1,*)|=\varepsilon m_0q_0$. Thus some point x_1 on the real line is contained in intervals for at least εm_0q_0 vertices of $C_0(1,*)$. We call the intervals containing the point x_1 a **stack** and we denote the collection $S(x_1)$. If a vertex in $C_0(1,*)$ has an interval in the stack $S(x_1)$ then we call that interval **primary** and the remaining $t-1$ intervals assigned to it **secondary**. Since εm_0q_0 of the vertices in $C_0(1,*)$ have a (primary) interval in the stack $S(x_1)$, the lemma implies that at least δm_0 of the cliques $C_0(1,j)$ have at least δq_0 vertices with (primary) intervals in stack $S(x_1)$. We may assume, by

appropriate relabelling, that this is true for cliques $C_0(1,1),\ldots,C_0(1,\delta m_0)$.

We now restrict our attention to an induced subgraph G_1 of G_0. Put $m_1=\delta m_0$, $q_1=\delta q_0$ and $G_1=G(n,m_1,q_1)$. (We do not alter f, but merely focus on its representation of G_1.) By analogy we define $C_1(i,j)$, $C_1(i,*)$ and $C_1(*,j)$. Notice that by our analysis above all vertices in $C_1(1,*)$ have a (primary) interval containing x_1. We now repeat the above analysis for $C_1(2,*)$: Some point x_2 on the real line contains intervals $S(x_2)$ from at least $\varepsilon m_1 q_1$ of the vertices in $C_1(2,*)$ and so, we may assume that cliques $C_1(2,1),\ldots,C_1(2,\delta m_1)$ each have at least δq_1 vertices with (primary) intervals in stack $S(x_2)$. Put $m_2=\delta m_1$ and $q_2=\delta q_1$. We let $G_2=G(n,m_2,q_2)$ and note that it has the property that all vertices in $C_2(1,*)$ and $C_2(2,*)$ have primary intervals in stacks $S(x_1)$ and $S(x_2)$ respectively.

We now continue to define G_3, G_4, etc. After n iterations we have $m=m_n=\delta^n m_0$ and $q=q_n=\delta^n q_0$ and $G=G_n=G(n,m,q)$. For all $i=1,\ldots,n$ we have that all mq vertices in $C(i,*)=C_n(i,*)$ have primary intervals containing the point x_i.

Suppose vertices v and w are in some clique $C(i,*)$. Thus their primary intervals intersect. By the non-redundancy assumption the $2(t-1)$ secondary intervals assigned to v and w are therefore disjoint.

Without loss of generality we may assume $x_1<x_2<\ldots<x_n$. One now checks that no primary interval containing x_i can contain $x_{i'}$ for any $i'\neq i$; otherwise some vertex in $C(i,j)$ (for some j) is adjacent to all vertices in $C(i',*)$, including those in $C(i',j')$ for some $j'\neq j$ (since $m=\delta^n m_0=2$, this is possible). This, however, is a contradiction since for no k,k' is $(i,j,k)\sim(i',j',k')$ when $i\neq i'$ and $j\neq j'$. It follows that if $|i-i'|>1$ then the primary intervals for (i,j,k) and (i',j',k') are

disjoint. In other words, primary intervals from non-consecutive stacks $S(x_i)$ and $S(x_{i'})$ cannot intersect.

We now consider the clique $C(*,1)$. It has nq vertices. By the usual argument there is a point y on the real line containing εnq intervals from $C(*,1)$. Moreover, at least δn of the cliques $C(i,1)$ have at least δq vertices in the stack $S(y)$. Note that $\delta n=3$. Thus there exist indices i and i' with $|i-i'|>1$ so that cliques $C(i,1)$ and $C(i',1)$ have at least $\delta q=\delta^{n+1}q_s=2$ intervals in stack $S(y)$. Since $\delta q=2>1$, these intervals must be primary because of the non-redundancy assumption. This, however, is a contradiction, since primary intervals from non-consecutive stacks are necessarily disjoint.∎

3.3.10 Remark. This result is best possible; $i(G)=1$ if and only if $NRI(G)=1$ since 1-interval representations are necessarily non-redundant. It should also be mentioned that if G is triangle-free, then $i(G)=NRI(G)$; this follows from the fact that any t-interval representation for G must be depth-2, and depth-2 representations can always be made non-redundant by "sliding" apart extra intersections. In general, one has $NRI(G)\le i_2(G)$. Thus by 2.1.20, one has $NRI(G)\le(\omega(G)-1)i(G)$.

4. IRREPRESENTABILITY

Every graph is the intersection graph of convex sets in $\mathbf{R}^3$. The same statement for $\mathbf{R}^2$, however, is false. Every graph is the intersection graph of curves in $\mathbf{R}^3$ but not in $\mathbf{R}^2$. Every graph is the intersection graph of boxes in $\mathbf{R}^\infty$ but not in any finite dimensional space. Thus there are cases when increasing the "dimension" of the sets we intersect results in our ability to represent all graphs. We consider, instead, the general quesion: when does increasing the **number** of sets we assign to each vertex enable us to represent all graphs? For example: is every graph the intersection graph of unions of two convex sets in $\mathbf{R}^2$?

We will completely answer the question of when increasing the number of sets assigned to a vertex results in the ability to represent all graphs. In other words, when does the S-number grow unbounded? We begin by showing in section 4.1 how some of these questions can be settled by using ad hoc arguments. We then give a more general theory of irrepresentability in section 4.2. This theory will give us all the same results we obtained by the methods of section 4.1 and additional results which are not readily obtainable by those methods. In section 4.3 we apply the theory in studying the intersection of line segments in vector spaces. In section 4.4 we compare our theory to a generalized theory for boxicity. We then generalize the results of 4.2 to nerves and simplicial complexes in section 4.5.

4.1 Ad Hoc Methods and Results

In this section we consider multiple intersection representations of boxes in finite dimensional space, as well as various types of connected sets in 2 dimensional manifolds. We demonstrate the existence of graphs which fail to be represented by any of the above methods.

4.1.1 Theorem. If t and d are positive integers then the class $\Omega(tB^d)$, the family of graphs representable by t boxes per vertex in $\mathbf{R}^d$, does not contain all graphs.

Proof. Let t and d be fixed. In every representation of a graph by boxes in $\mathbf{R}^d$ it is only the **order** of the endpoints of the defining intervals of the boxes which matters in determining which box intersects which. Thus if a graph has n vertices, we lose no generality in forming a t-box intersection representation for this graph if we assume that the coordinates of the corners of the boxes are integers in the set $\{1,2,\ldots,2nt\}$.

Suppose $\Omega(tB^d)$ contains all graphs. Let n be a positive integer. The number of labeled graphs with n vertices is $2^{\frac{1}{2}n(n-1)}$ and so the number of non-isomorphic graphs on n vertices is at least $2^{\frac{1}{2}n(n-1)}/n! \geq n^{-n}2^{\frac{1}{2}n(n-1)}$.

The number of boxes in $\mathbf{R}^d$ with corners in the set $\{1,2,\ldots,2nt\}^d$ is at most $(2nt)^{2d}$ and therefore the number of nonisomorphic graphs with n vertices in $\Omega(tB^d)$ is at most $[(2nt)^{2d}]^{nt} = (2nt)^{2dnt}$. It follows that

$$(2nt)^{2ndt} \geq n^{-n}2^{\frac{1}{2}n(n-1)}$$

or

$$n^n(2nt)^{2ndt} \geq 2^{\frac{1}{2}n(n-1)}.$$

Taking logarithms to the base 2 we have

$$n(\log n) + 2ndt(\log 2nt) \geq \tfrac{1}{2}n(n-1)$$

or

$$(1+2td)(\log n) + 2td(\log 2t) \geq \tfrac{1}{2}(n-1).$$

This last inequality is obviously false for n sufficiently large.∎

4.1.2 Remark. One corollary of the above proof is that there exist graphs G for which i(G) is arbitrarily large. This is no surprise since we know that the interval number of $K_{n,n}$ is roughly $\frac{1}{2}n$.

The above proof can be paraphrased in terms of "information". A graph on n vertices contains roughly $\frac{1}{2}n^2$ bits of information while a t-box representation in $\mathbf{R}^d$ contains (roughly) a constant times n(log n) bits of information. [The nt boxes are each specified by 2d integers that are all between 1 and 2nt. An integer M requires log(M) bits.] Since n^2 grows much faster than n(log n), we see that the result must follow.

The proof has a probability interpretation: If we denote the family ofall graphs on at most n vertices by $\mathbf{G}(n)$ and the probability that a random graph in $\mathbf{G}(n)$ is in $\Omega(t\mathbf{B}^d)$ by P(n,t,d) then

$$P(n,t,d) \le [2^{\frac{1}{2}n(n-1)}]/[n!(2nt)^{2dnt}].$$

It follows that as $n\to\infty$, $P(n,t,d)\to 0$. Thus "almost all graphs" are not in $\Omega(t\mathbf{B}^d)$.

We can also use the inequalities in the proof to show

$$\sup\{\mathbf{B}^d\#(G): |V(G)|\le n\} \ge (n/(8d \log n)) - (1/2d).$$

This estimate is probably much to low since for d=1 we have by 2.2.1

$$\sup\{i(G): |V(G)|\le n\} = \lceil \tfrac{1}{4}(n+1)\rceil.$$

The primary advantage to the above proof is its simplicity. The primary drawback is that it is non-constructive.

We now turn to study irrepresentability by multiple intersection of various "two dimensional" objects. Our study is motivated by the question: "Does every graph possess an intersection representation in which each vertex is assigned two convex sets in $\mathbf{R}^2$?" As we shall see the answer to this question is "no" and remains "no" even when "two" is

replaced by any other positive integer and when the words "convex sets in $\mathbf{R}^2$" are replaced by "arc-connected sets in a 2-dimensional manifold with finite Euler characteristic." Our aim is to prove:

Theorem (4.1.13). Let t be a positive integer and M a two-dimensional manifold with finite Euler characteristic. There exists a graph G which is not the intersection graph of sets each of which is the union of (up to) t connected compact [or connected open, or arc-connected] subsets of M.

Our method is constructive but tedious. A more elegant but non-constructive approach will be presented in the next section.

We will obtain our results by "pulling ourselves up by our bootstraps". Our first result will be highly specialized and we use it to obtain increasingly general theorems. We begin with some preparatory material.

4.1.3 Lemma. Let M be a 2-manifold with Euler characteristic χ. If G embeds in M then $|E(G)|\le 3(|V(G)|-\chi)$.

Proof. This result is well known and is obtained from the definition of the Euler characteristic by noting that each face of the embedded graph has at least three sides.∎

4.1.4 Definition. A **piecewise linear** curve, or PL curve, is a function $f:[0,1]\to V$, with V a vector space, with the property that there exist numbers $0=s_0<s_1<\ldots<s_p=1$ such that $f(\lambda s_i+(1-\lambda)s_{i+1}) = \lambda f(s_i)+(1-\lambda)f(s_{i+1})$ for $0\le\lambda\le 1$ and $0\le i\le p-1$. In other words, $f([s_i,s_{i+1}])$ is a line segment in V. We usually identify a PL curve with its image $f([0,1])\subset V$. In other words, a PL curve is a curve which is the union of line segments.

A 2-dimensional manifold is **piecewise linear** if it is the union of flat 2-dimensional geometric simplices, i.e. "filled in" triangles.

4.1.5 Lemma. If C is a non-self-intersecting curve on a 2-manifold M, then there exists a positive number ε such that the set off all points of M within distance less than ε of C (the ε-neighborhood of C) is homeomorphic to the interior of a disc.∎

4.1.6 Theorem. Let t be a positive integer and M be a PL 2-manifold with Euler characteristic χ. There exists a graph which is not the intersection graph of sets each of which is the union of t PL curves in M such that the PL curves satisfy:

(1) no PL curve intersects itself,

(2) no two PL curves assigned to the same vertex intersect,

(3) if PL curves intersect, the intersection consists of finitely many points, and

(4) no three PL curves share a common point.

The PL restrictions and (1)-(4) are included to avoid topological difficulties. After proving this theorem we will show how all of these restriction may be dropped.

Proof. Choose a positive integer N large enough so that

(*) $\binom{N}{2} > 3(Nt - \chi)$.

Let $n = r(t+1,N)$. [See §0.5.2 for the definition of the Ramsey number r.] We define a bipartite graph G with $n + \binom{n}{t+1}$ vertices as follows: The vertices in the first part are $\{v_1,\ldots,v_n\}$. The vertices in the second part are:

$\{w_I\colon I\subset\{1,2,\ldots,n\}$ and $|I|=t+1\}$.

Thus the vertices in the second part are indexed by the subsets of $\{1,\ldots,n\}$ of cardinality t+1. Finally v_i is adjacent to w_I if and only if $i\in I$. This completes the definition of G. We claim that G does not

have an intersection representation satisfying the conditions of the theorem.

[Example: When $t=1$ and $\chi=2$ then $N=5$ satisfies (*) and $n=r(2,5)=5$. The graph G constructed above is precisely βK_5.]

Suppose G had a representation by PL curves as described in the theorem. Let the PL curves assigned to vertex v_i be $v_i^1,\ldots,v_i^t$ and the curves assigned to w_I be $w_I^1,\ldots,w_I^t$.

We say the curve w_I^s joins curves v_i^p and v_j^q, where $i\neq j$ and $i,j\in I$, provided there exists numbers $x,y\in[0,1]$ so that $w_I^s(x)$ is on curve v_i^p and $w_I^s(y)$ is on v_j^q and if z is between x and y then $w_I^s(z)$ is on no other curve. In other words, one can traverse curve w_I^s from v_i^p to v_j^q without meeting any other curves. In figure 4.1 w_I^2 joins

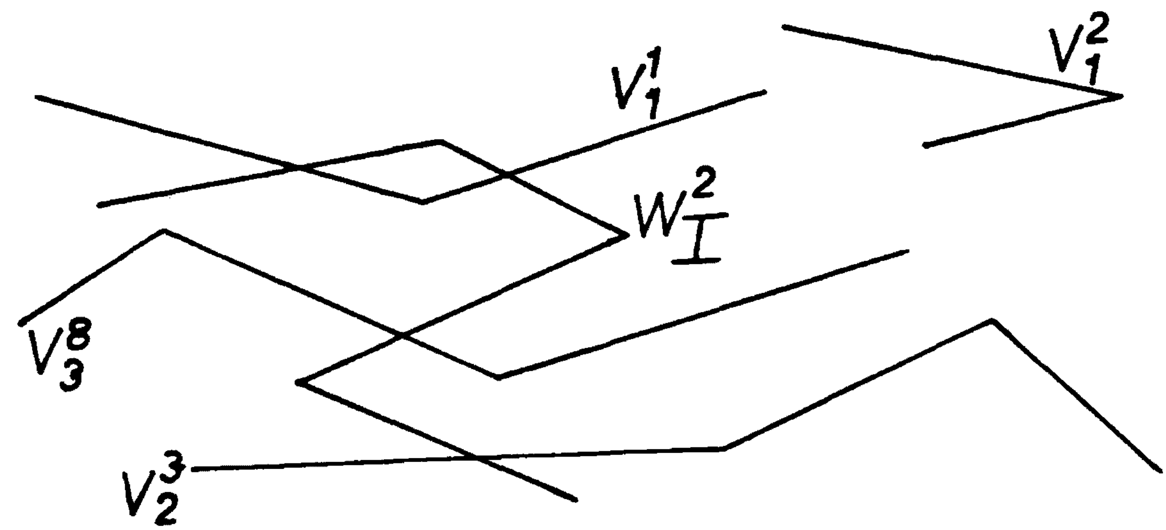


Figure 4.1

v_1^1 to v_3^8, as well as v_3^8 to v_2^3.

Consider the graph K_n whose vertices are $\{u_1,\ldots,u_n\}$. Color edge

u_iu_j red if some v_i^p is joined to some v_j^q by some w-curve. Color edge u_iu_j blue otherwise. We claim there is no blue (t+1)-clique in this K_n.

Suppose $u_{i_1},\dots,u_{i_{t+1}}$ formed a blue clique. Let $I=\{i_1,\dots,i_{t+1}\}$. Consider the t curves $w_I^1,\dots,w_I^t$. At least one of these w-curves must meet curves from two of $v_{i_1},\dots,v_{i_{t+1}}$. This follows from the pigeon hole priniciple since there are t curves w_I^s meeting curves from t+1 vertices. Suppose w_I^1 meets curves from two different vertices. As we traverse w_I^1 we encounter finitely many points of intersection with v-curves. It follows that some sequential pair of points of intersection must belong to v_j^p and v_k^q with $j\neq k$, $j,k\in I$. Thus u_ju_k would be colored red.

Since $n=r(t+1,N)$ and there is no blue (t+1)-clique in this K_n, we must have a red N-clique. We may assume that the red clique consists of the vertices: $u_1,\dots,u_N$. We now consider a third graph H. The vertices of H are the tN curves v_i^p with $1\le p\le t$ and $1\le i\le N$. Vertices of H are adjacent if and only if the curves are joined by some w_I^s. Clearly, $|E(H)|\ge\binom{N}{2}$ since $\{u_1,\dots,u_N\}$ forms a red clique. We now embed H in M as follows:

For each curve v_i^p we choose an ε such that the ε-neighborhood in M of this curve is homeomorphic to a disc and so that no two ε-neighborhoods intersect. [This is possible since by hypothesis the curves v_i^p do not self-intersect and are disjoint compacta.] Replace the curve v_i^p with a single point inside its ε-neighborhood. If a w_I^s meets v_i^p, and serves to join this curve to another v-curve, we replace the portion of the curve w_I^s inside the ε-neighborhood with a path linking the curve with the chosen point. See figure 4.2. We can do this in such a way that distinct altered w-curves meet only at the selected points inside the ε-neighborhoods. Clearly we have embedded the graph H in the 2-manifold M with the selected points as topological vertices and

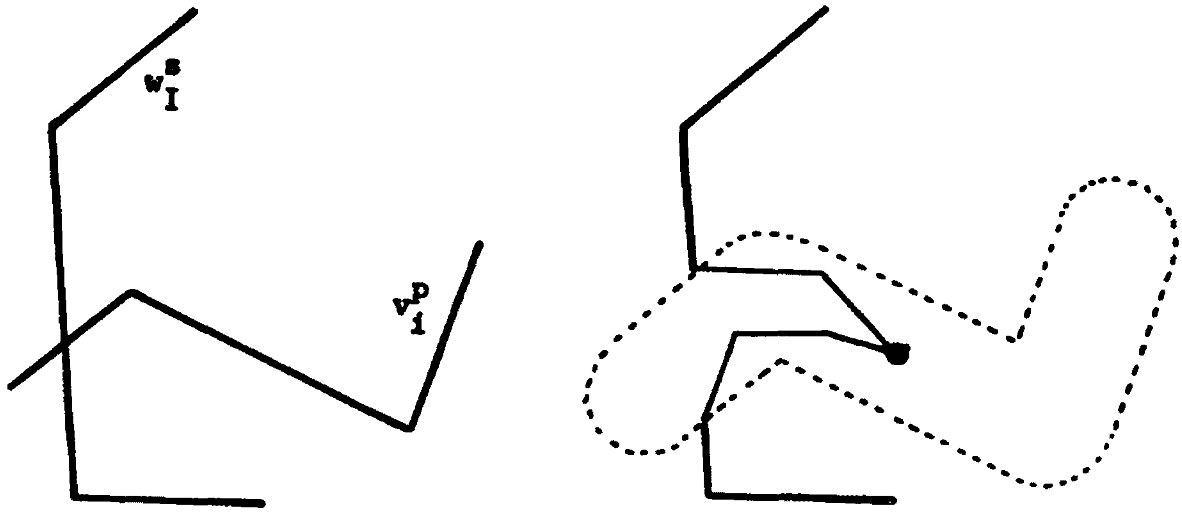


Figure 4.2

the altered w-curves forming the topological edges. It follows by proposition 4.1.3 that $|E(H)|\leq 3(|V(H)|-\chi)$. Hence $\binom{N}{2}\leq 3(Nt-\chi)$. But this contradicts (*). Hence the graph G cannot have a representation of the form stated.∎

We now proceed to show that the theorem is true as we successively remove the "artificial" restrictions in the hypothesis.

4.1.7 Proposition. Hypothesis (4) can be dropped from Theorem 4.1.6.

Proof. Suppose G is a graph whose t PL curve representation satisfies conditions (1), (2) and (3). We show that small alterations in that representation form a representation that satisfies (1) through (4). For every three curves in the representation of G there are at most finitely many points of M in all three curves. Choose a small neighborhood in M of each of these points homeomorphic to a disc. We may assume that these neighborhoods are disjoint. We now slightly alter one of these curves near each triple point (inside the disc) so that no

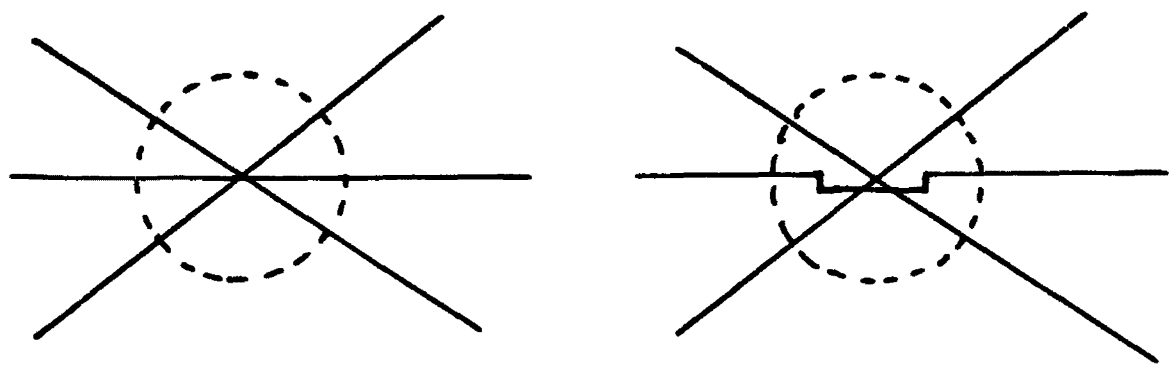


Figure 4.3

intersections are lost or added. See figure 4.3. Repeating this process for all triples of curves we obtain a representation for G satisfying (1) though (4).∎

4.1.8 Proposition. Hypotheses (1) and (4) may be dropped from Theorem 4.1.6.

Proof. Suppose G has a t PL curve representation in M satisfying (2) and (3). If a PL curve intersects itself, it does so in finitely many points by (3). Select disjoint neighborhoods in M homeomorphic to discs about each self-intersection point of each PL curve. Alter the curve as in figure 4.4 or 4.5 depending upon whether or not the self-intersection is a crossing. If the self-intersection is a crossing we reconnect so that the altered curve is still in one piece. We can do this without disturbing any intersections between pairs of curves. The resulting representation for G satisfies (1), (2) and (3) and we apply 4.1.7 to complete the proof.∎

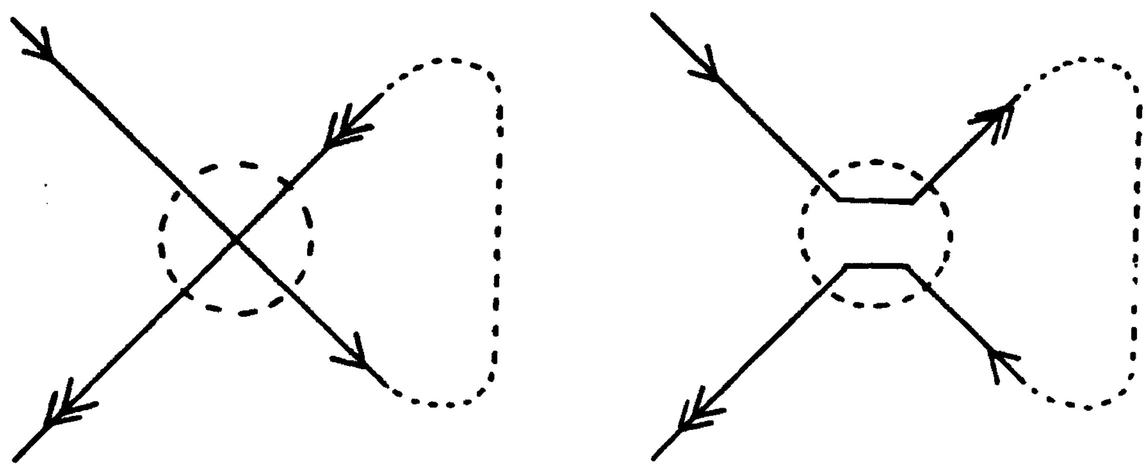


Figure 4.4

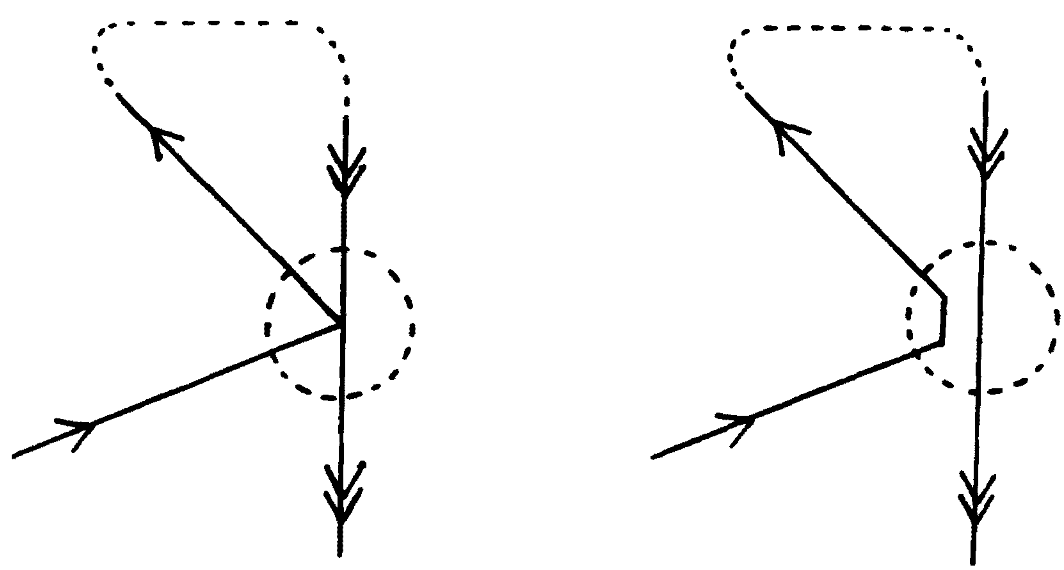


Figure 4.5

4.1.9 Proposition. Hypotheses (1), (2) and (4) may be dropped from Theorem 4.1.6.

Proof. Suppose G has a t PL curve representation in M satisfying (3). If two PL curves assigned to the same vertex intersect we alter the

curves as in the last proof. See figure 4.6. This depends only on

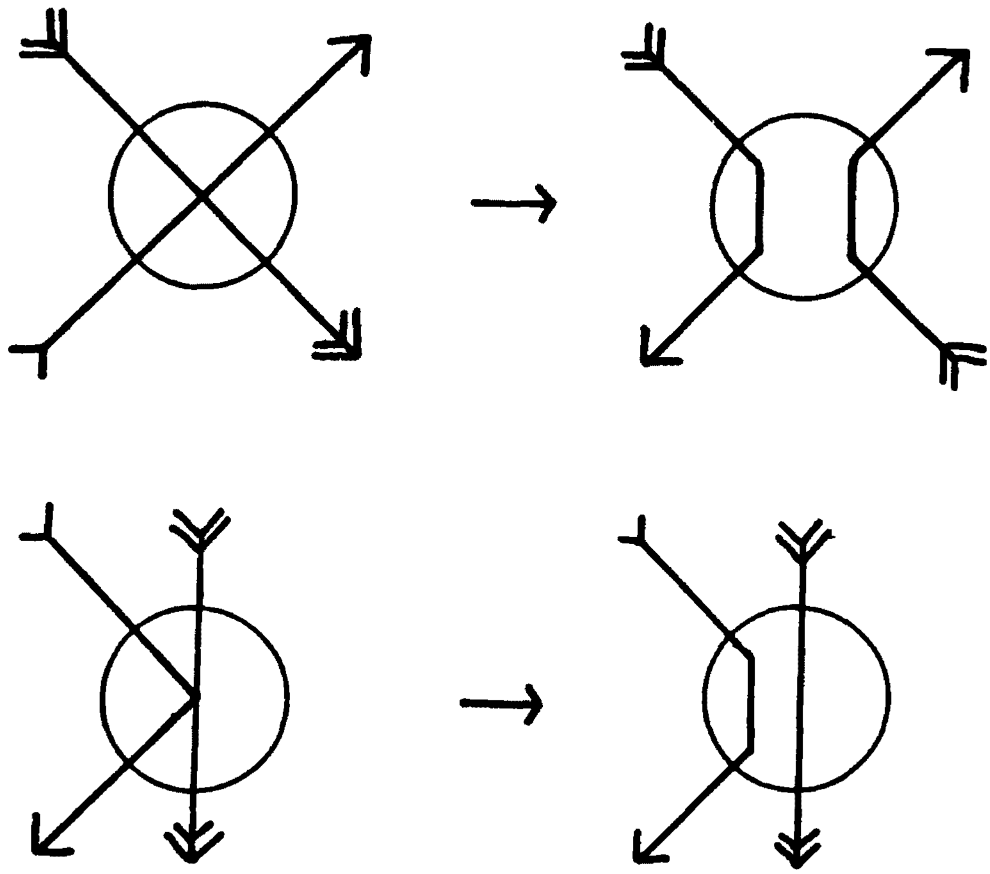


Figure 4.6

whether the two curves cross or not. The representation now satisfies (2) and (3) and the result follows by 4.1.8.∎

4.1.10 Proposition. Hypotheses (1) through (4) may be dropped from Theorem 4.1.6.

Proof. In light of 4.1.9 it is enough to show how if two PL curves (or a curve and itself) of a representation have infinite intersection, then they can be altered so as to maintain all intersections such that the resulting curves have finite intersections.

If two PL curves have infinite intersection, then that intersection is the union of finitely many line segments. For each of

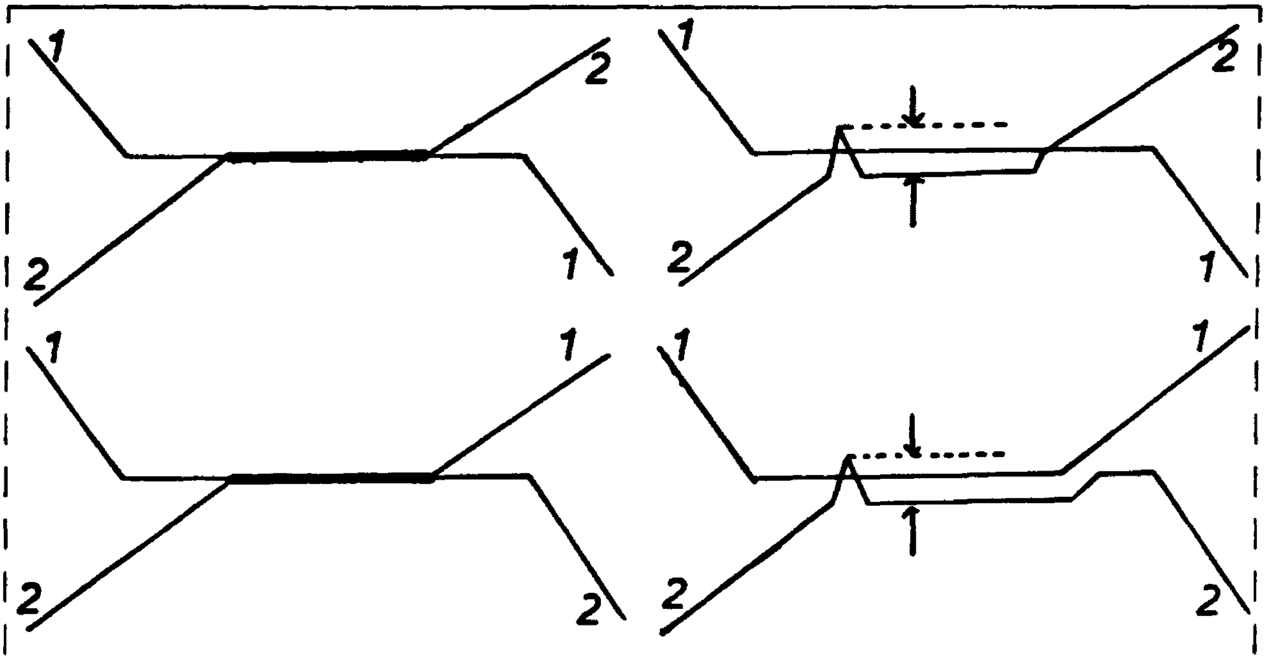


Figure 4.7

these line segments we alter the PL curve as in figure 4.7. Notice that the "blips" assure us that no intersections were lost. We can be sure no new intersections are formed if we follow the following rule: Since the curves are compact there exists a positive number ε such that the minimum distance between nonintersecting curves is ε. At the n^{th} alteration of the representation, move no point more than a distance $2^{-n}\varepsilon$. ∎

We now gradually remove the PL restrictions in Theorem 4.1.6.

4.1.11 Theorem. Let t be a positive integer and M be a PL 2-manifold with Euler characteristic χ. There exists a graph which is not the intersection graph of sets each of which is the union of t open connected subsets of M.

Proof. It is well known that open connected subsets of a manifold are arc-connected and we may assume that the arcs are PL since the 2-manifold M is PL. Let G be represented as in the hypothesis of the theorem. Let the open sets be $O_1, O_2, O_3, \ldots$. Choose a point $x_i \epsilon O_i$ and if O_i and O_j intersect, choose a point $y_{ij} \epsilon O_i \cap O_j$. [Note: $y_{ij} = y_{ji}$.] For each i, join x_i to y_{ij} by a PL curve contained in O_i. Notice that we can consider the curves radiating from x_i together as one large (possibly self-intersecting) curve by "running" up and down each "branch". Notice further that this gives a t PL curve representation of G since if two O's intersect then the corresponding curves must both contain the same point y and if two O's do not intersect, then the curves they contain cannot intersect either. The result now follows from 4.1.10.∎

4.1.12 Theorem. Let t be a positive integer and M be a PL 2-manifold with Euler characteristic χ. There exists a graph which is not the intersection graph of sets each of which is the union of t connected compact subsets of M.

Proof. Suppose G has an intersection representation with t connected compact subsets of M assigned per vertex. Let the compact subsets be $C_1, C_2, C_3, \ldots$. Let ε be a positive number less than the minimum distance between non-intersecting C_i's. Let O_i be the set of all points of M whose distance from C_i is less than $\frac{1}{2}\varepsilon$. Clearly O_i is open. If we choose ε small enough, O_i is connected. It is now obvious that replacing the C_i's by O_i's gives a representation of G and the result follows by 4.1.11.∎

4.1.13 Theorem. Let t be a positive integer and let M be (any) 2 dimensional manifold with Euler characteristic χ. There exists a graph which is not the intersection graph of sets each of which is the union of up to t connected compact [open] [arc-connected] subsets of M.

Proof. Since every manifold is homeomorphic to a PL manifold and since homeomorphism preserves open sets as well as compact sets, the result follows for compact and open subsets from 4.1.12. In case the subsets are arc-connected we perform the same construction as in the proof of theorem 4.1.11 to show that any intersection graph formed by arc-connected sets can equivalently be formed by curves, which are compact.∎

4.1.14 Corollary. Let t be a positive integer. There exists a graph which is not the intersection graph of sets each of which is the union of t convex sets [resp. curves] in the plane.

Proof. The plane is a 2-manifold. Curves are compact and convex sets are arc-connected.∎

4.2 General Theory

In the last section we saw that not every graph could be represented by fixed multiple intersection of boxes in finite dimensional space or of various two dimensional connected sets. However, the techniques of the last section shed no light on the question: "given $t>0$, is every graph the intersection graph of sets each of which is the union of t line segments in a vector space?" The method of Theorem 4.1.1 will not work because more than the order of the endpoints is important (see figure 4.8). Nor will the method of Theorem 4.1.6 help since the segments are not constrained to lie in a two-dimensional structure. Thus in order to answer this question we need more powerful techniques. We begin by broadening the S-number concept to the "P-number", where **P** is any class of graphs. Should $\mathbf{P}=\Omega(\mathbf{S})$ we will see that $\mathbf{P}\#=\mathbf{S}\#$. We then analyze the boundedness of the P-number parameter.

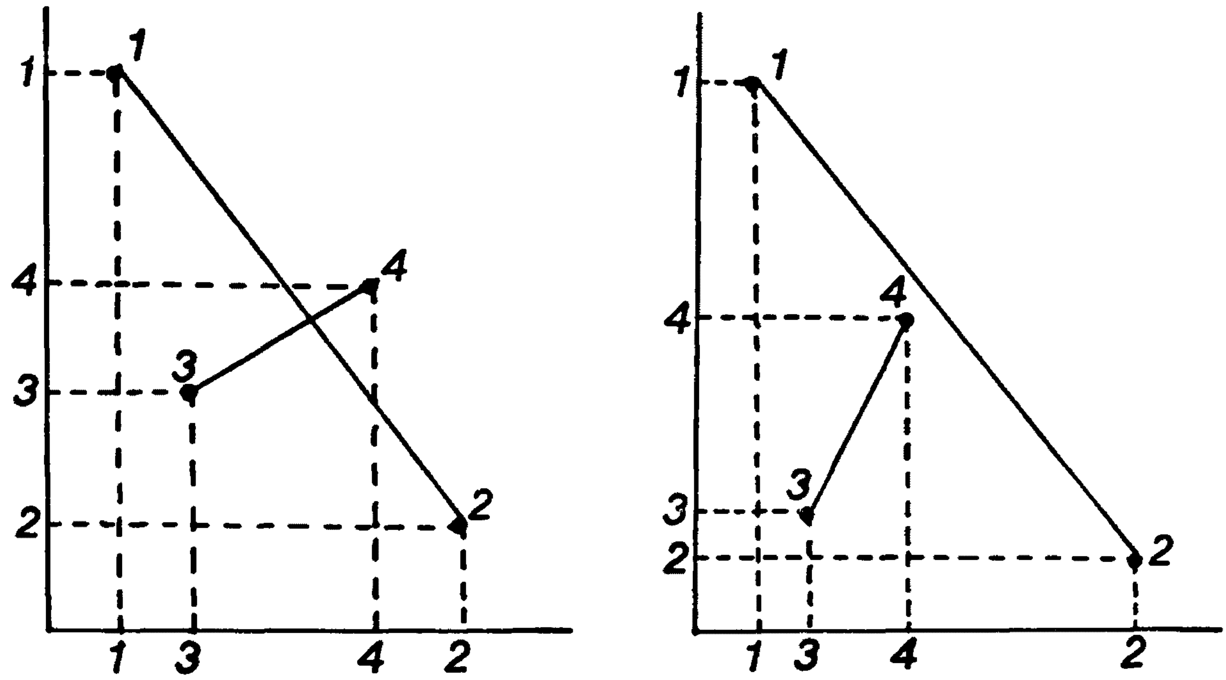


Figure 4.8

4.2.1 Definition. Let A,B be finite sets and let $f:A\to B$ be onto. The **multiplicity** of f, denoted mult(f), is defined to be

$$\text{mult}(f) = \max\{|f^{-1}(b)|:\ b\in B\}.$$

Notice that if mult(f) = 1, then f is a bijection.

4.2.2 Definition. Let G and H be graphs and let $f:V(G)\to V(H)$. We say that f is a **simplicial map** provided f satisfies:

(1) f is onto,

(2) If $w_1,w_2\in V(H)$ and $w_1\sim w_2$, then there exist $v_1\in f^{-1}(w_1)$ and $v_2\in f^{-1}(w_2)$ such that $v_1\sim v_2$, and

(3) If $v_1,v_2\in V(G)$, $v_1\sim v_2$ and $f(v_1)\neq f(v_2)$, then $f(v_1)\sim f(v_2)$.

In other words, f is simplicial means xy is an edge in H if and only if there is an edge joining some vertex in the pre-image of x to some vertex in the pre-image of y. Notice that we "ignore" edges joining vertices in the pre-image of any single vertex. Also note that if the multiplicity of simplicial map is 1, then it is a graph isomorphism.

4.2.3 Definition. Let **P** be a family of graphs and let G be a graph. The **P-number** of G, denoted **P**#(G), is defined to be the least positive integer t such that there exists a graph H$\in$**P** and a simplicial map f:V(H)$\to$V(G) such that mult(f)=t. In case no such map exists we put **P**#(G)=∞. The **P**-number is an extension of the **S**-number:

4.2.4 Proposition. If **P** is an intersection class of graphs with **P**=Ω(**S**), then **P**#(G)=**S**#(G) for all graphs G.

Proof. Let G be a graph. Suppose G$\in\Omega$(t**S**) for an integer t. Let V(G) = $\{v_1,\ldots,v_n\}$ and let g:V(G)$\to$t**S** be a t**S**-representation. Thus $g(v_i) = S_i^1\cup\ldots\cup S_i^t$ for $S_i^p\in$**S**. Let $H = \Omega[S_i^p\text{: } 1\le i\le n \text{ and } 1\le p\le t]$. Clearly H$\in$**P**. Let f:V(H)$\to$V(G) be defined as follows: if w_i^p is the vertex assigned to S_i^p then let $f(w_i^p)=v_i$. Clearly $v_i\sim v_j$ provided some S_i^p meets some S_j^q, i.e. if and only if $w_i^p\sim w_j^q$ for some $1\le p,q\le t$. Thus f is simplicial. Clearly mult(f)$\le$t. Therefore **P**#(G)$\le$**S**#(G).

Conversely, suppose we have a simplicial map f:V(H)$\to$V(G) with mult(f)=t and H$\in$**P**. Let g:V(H)$\to$**S** be an **S**-representation of H. Let a function h be defined on V(G) by h(v)=$\cup$g(w) with the union taken over all w$\in f^{-1}(v)$. Since $|f^{-1}(v)|\le t$ it follows that h:V(G)$\to$t**S**. Further for v,v'$\in$V(G) we see that v$\sim$v' if and only if for some w$\in f^{-1}(v)$ and w'$\in f^{-1}(v')$ we have w$\sim$w', which holds if and only if g(w)$\cap$g(w')$\neq\emptyset$. It follows that v$\sim$v' if and only if h(v)$\cap$h(v')$\neq\emptyset$. Hence G$\in\Omega$(t**S**). Therefore **S**#(G)$\le$**P**#(G).

Note that we have shown that if either **S**#(G) or **P**#(G) is finite, then so is the other. Thus if one is infinite, the other must be also.∎

It is now immediate that if Ω(**S**)=Ω(**S'**), then **S**#=**S'**#. In addition, we will no longer distinguish between **P**# and **S**# when **P**=Ω(**S**).

Recall that if f is a simplicial map and mult(f)=1 then f is a graph isomorphism. This yields the following:

4.2.5 Proposition. $P\#(G) = 1$ if and only if $G \in P$.∎

4.2.6 Remark. We turn now to the question of when the P# is bounded. We begin by showing when the P# is "well-defined," i.e. finite for all graphs. For example, if P is the family of all complete graphs, then if G is not complete, $P\#(G)=\infty$.

Although we have defined P# for arbitrary classes of graphs P, we can say very little of interest about these parameters if we allow P to be arbitrary. If we impose the restriction that P be monotone, we can obtain interesting results. In as much as we are interested in the P# in order to shed light on multiple intersection, and since intersection classes are necessarily monotone, requiring P to be monotone is not a serious limitation for our purposes.

4.2.7 Proposition. If P is a monotone class of graphs, then $P\#(G)$ is finite for all graphs G if and only if $nK_2 \in P$ for all n, in which case $P\#(G) \le \Delta(G)$.

Proof. Suppose for some n the class P does not contain nK_2. We show $P\#(nK_2)=\infty$. Suppose $P\#(nK_2) = t < \infty$. Let $f:V(H) \to V(nK_2)$ be a simplicial map with $H \in P$. Let $V(nK_2)=\{v_1,\dots,v_n,w_1,\dots,w_n\}$ and $f^{-1}(v_i)=\{v_i^1,v_i^2,\dots\}$ and $f^{-1}(w_i)=\{w_i^1,w_i^2,\dots\}$. We may relabel the superscripts so that $v_i^1 \sim w_i^1$ for $1 \le i \le n$. Notice that:

v_i^p is not adjacent to v_j^q for $i \ne j$,

v_i^p is not adjacent to w_j^q for $i \ne j$, and

w_i^p is not adjacent to w_j^q for $i \ne j$.

It follows that the induced subgraph of H on vertices with superscript 1 is isomorphic to nK_2 and since $nK_2 \le H \in P$ and P is monotone, $nK_2 \in P$ which

is impossible. Thus $P\#(nK_2)=\infty$.

Conversely, suppose $nK_2 \in P$ for all n and let G be a graph. Let $\Delta=\Delta(G)$ be the maximum degree of a vertex in G. We show that $P\#(G)\le\Delta$. Notice that P contains all graphs of the form $aK_2+bK_1 \le (a+b)K_2 \in P$ since P is monotone. We now define a graph, H, with $\Delta|V(G)|$ vertices. If $V(G)=\{v_1,\dots,v_n\}$, let $V(H) = \{v_i^p : 1\le i\le n \text{ and } 1\le p\le\Delta\}$. Next define edges in H as follows:

Consider the edges of G in some order. For each v_iv_j of G place an edge between (until now) isolated vertices v_i^p and v_j^q of H. Since no vertex of G has more incident edges than it has copies in H, there are always enough vertices available. The resulting graph H has $\Delta(H)\le 1$ and therefore is of the form aK_2+bK_1 and so $H\in P$. Let $f:V(H)\to V(G)$ by $f(v_i^j)=v_i$. It is now clear that f is simplicial and $\mathrm{mult}(f)\le\Delta$.∎

4.2.8 Corollary. Let P be the class of graphs $\{aK_2+bK_1 : a,b\ge 0\}$ and let G be a graph. Then $P\#(G) = \Delta(G)$.

Proof. Clearly P is monotone, and by the previous proof $P\#(G)\le\Delta(G)$. Suppose $H\in P$ and $f:V(H)\to V(G)$ is simplicial. Let v be a vertex of G with $d(v)=\Delta(G)$. Since the number of edges of H entering $f^{-1}(v)$ must be at least $\Delta(G)$ and $\Delta(H)\le 1$, it follows that $|f^{-1}(v)|\ge\Delta(G)$. Thus $\mathrm{mult}(f)\ge\Delta(G)$.∎

4.2.9 Remark. If P is a monotone class for which $P\#$ is always finite then $P\#(G)\le\Delta(G)$ and this is the "best" uniform upper bound. However, for most applications this upper bound is very far from being the optimal. In §2.2.3 we saw that $i(G)\le\lceil\frac{1}{2}(\Delta(G)+1)\rceil$ and if P is any intersection class of nonempty sets, then $P\#(K_n)=1\ll\Delta(K_n)=n-1$.

We now address the following question: For which monotone classes

P does there exist a number N so that $\mathbf{P}\#(G)\le N$ for all graphs G? The answer:

4.2.10 Theorem. Let **P** be a monotone class of graphs, and let $N=\sup\{\mathbf{P}\#(G):G\in\mathbf{G}\}$. Then N equals:

1, if **P** contains all graphs, or

2, if **P** contains all bipartite graphs, but not all graphs, or

∞, if **P** does not contain all bipartite graphs.

Proof. Let $N=\sup\{\mathbf{P}\#(G):G\in\mathbf{G}\}$. If $\mathbf{P}=\mathbf{G}$, then by 4.2.5 $\mathbf{P}\#(G)=1$ for all graphs G and N=1.

Suppose **P** contains all bipartite graphs, but not all graphs. If G is not in **P**, then $\mathbf{P}\#(G)>1$ by 4.2.5, hence N>1. Let $V(G)=\{v_1,\dots,v_n\}$ and we define a new graph H with $V(H)=\{x_1,\dots,x_n,y_1,\dots,y_n\}$, and $E(H)=\{x_iy_j:v_iv_j\in E(G)\}$. Clearly the x's and y's form independent sets, and therefore H is bipartite and by hypothesis $H\in\mathbf{P}$. Let $f:V(H)\to V(G)$ by $f(x_i)=f(y_i)=v_i$. Clearly f is simplicial with multiplicity 2, hence $\mathbf{P}\#(G)\le 2$ for all graphs. Thus N=2.

Suppose **P** does not contain a bipartite graph, G. Let t be a positive integer, and let $T=t^2$. By the bipartite version of the edge induced Ramsey theorem (§§0.5.3-4) there exists a bipartite graph G' such that for every T coloring of the edges of G' there exists an induced copy of G in G' in which all edges have the same color. We show that $\mathbf{P}\#(G')>t$.

Suppose $\mathbf{P}\#(G')\le t$ and let $f:V(H)\to V(G')$ be a simplicial map of multiplicity at most t with $H\in\mathbf{P}$. Since G' is bipartite let $V(G')=X\cup Y$ be a partition of the vertices of G' into independent sets. For $x_i\in X$ let $f^{-1}(x_i)=\{x_i^1,\dots,x_i^t\}$, and for $y_i\in Y$ let $f^{-1}(y_i)=\{y_i^1,\dots,y_i^t\}$. If $x_iy_j\in E(G')$ then some x_i^p is adjacent to y_j^q for some (p,q). Color edge

x_iy_j with color (p,q). [In case there is more than one choice, select one of those colors arbitrarily.] This gives a $T=t^2$-coloring of the edges of G'.

By the way the graph G' was defined, there exists a monochromatic induced copy of G in G'. Let the color be (p,q). We may assume, without loss of generality, that the (p,q)-color induced copy of G in G' occurs on vertices $x_1,\ldots,x_n,y_1,\ldots,y_m$. Consider the induced subgraph H' of H on vertices $x_1^p,\ldots,x_n^p,y_1^q,\ldots,y_m^q$. We claim that H' is isomorphic to G, indeed the isomorphism is $f|V(H')$. First, it is clear that $f|V(H')$ is a bijection: $x_i \leftrightarrow x_i^p$ and $y_j \leftrightarrow y_j^q$. Second, if $x_i \sim y_j$ then, since x_iy_j is colored (p,q) we know that $x_i^p \sim y_j^q$. Conversely, if $x_i^p \sim y_j^q$, then $x_i \sim y_j$ since f is simplicial. Thus H' is isomorphic to G. However $H' \leq H \in P$ and P is monotone, hence $G \in P$, contrary to our supposition. Thus $P\#(G)>t$, and since t was arbitrary, $N = \infty$.∎

Thus the P-number grows unbounded if and only if P fails to contain all bipartite graphs. In the next section we apply this fact to various intersection classes.

4.3 Applications, Line Segments

We begin by showing how we can give easier proofs for the results on two-dimensional sets proved in section 4.1.

4.3.1 Theorem. Let t be a positive integer and let M be a 2-manifold with Euler characteristic χ. Then there exists a graph G such that G has no intersection representation by sets each of which is the union of up to t:

(1) curves in M,

(2) connected compact sets in M,

(3) arc-connected sets in M, or

(4) convex sets in the plane.

Indication of Proof. In as much as this theorem was carefully proved in section 4.1 we give only a proof outline, omitting the topological details.

Let **S** be one of the family of sets described in (1) through (4) and let $P=\Omega(S)$. Choose n sufficiently large so that $\binom{n}{2}>3(n-\chi)$. Thus K_n does not embed in M. We claim that the subdivision graph βK_n is not a member of **P**. Since βK_n is bipartite, 4.2.10 implies we can find a graph for which $P\#(G)>t$.

Suppose $\beta K_n \epsilon \Omega(S)$. Choose an S-representation for βK_n has n vertices of degree n-1 which we call "vertex-like" and $\binom{n}{2}$ vertices of degree 2 which we call "edge-like". Choose a point in each set **S** assigned to a vertex-like vertex. These points can then be joined, one to the other, by curves contained (essentially) in the sets assigned to the edge-like vertices, resulting in an embedding of K_n in M, which is impossible.∎

4.3.2 Remark. We can also prove, using Theorem 4.2.10, that not every graph has an intersection by t boxes in d-dimensional space by showing there exist bipartite graphs with arbitrarily high boxicity. This can be done via a counting argument akin to the one in §4.1.1 by observing that the number or non-isomorphic bipartite graphs on n vertices exceeds $2^{\frac{1}{4}n^2}/n^n$ but the number of non-isomorphic graphs with bounded boxicity and n vertices is less than (roughly speaking) a constant times n^n. Taking logarithms we see that

$$\tfrac{1}{4}n^2 - n(\log n) \gg n(\log n)$$

and the result follows.

We now study the intersection graphs of line segments in Euclidean

space.

4.3.3 Notation. Let $\mathbf{LS}^d$ denote the set of all line segments in $\mathbf{R}^d$. Let $\mathbf{LS}^\infty$ denote the set of all line segments in an infinite dimensional real vector space.

4.3.4 Proposition. $\Omega(\mathbf{LS}^1)\subset\Omega(\mathbf{LS}^2)\subset\Omega(\mathbf{LS}^3)\subset\ldots$ and $\Omega(\mathbf{LS}^\infty)=\cup_{i>0}\Omega(\mathbf{LS}^i)$.

Proof. This is trivial since we can consider $\mathbf{LS}^1\subset\mathbf{LS}^2\subset\ldots$ and if $G\in\Omega(\mathbf{LS}^\infty)$ and $|V(G)|=n$, then clearly $G\in\Omega(\mathbf{LS}^{2n})$ by considering the subspace spanned by the 2n endpoints of the line segments assigned to the vertices of G.∎

We now consider which of the inclusions $\Omega(\mathbf{LS}^i)\subset\Omega(\mathbf{LS}^{i+1})$ are proper.

4.3.5 Theorem. $\Omega(\mathbf{LS}^1)\neq\Omega(\mathbf{LS}^2)\neq\Omega(\mathbf{LS}^3)$, but $\Omega(\mathbf{LS}^n)=\Omega(\mathbf{LS}^{n+1})$ for $n\geq3$, thus $\Omega(\mathbf{LS}^3)=\Omega(\mathbf{LS}^\infty)$.

Proof. $\Omega(\mathbf{LS}^1)$ is the class of interval graphs and C_4 is not an

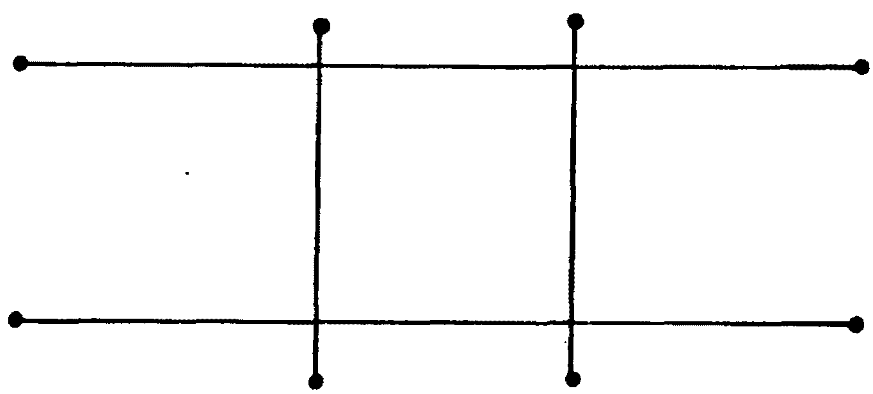


Figure 4.9

interval graph. However $C_4\in\Omega(\mathbf{LS}^2)$, as shown in figure 4.9. βK_5 is not

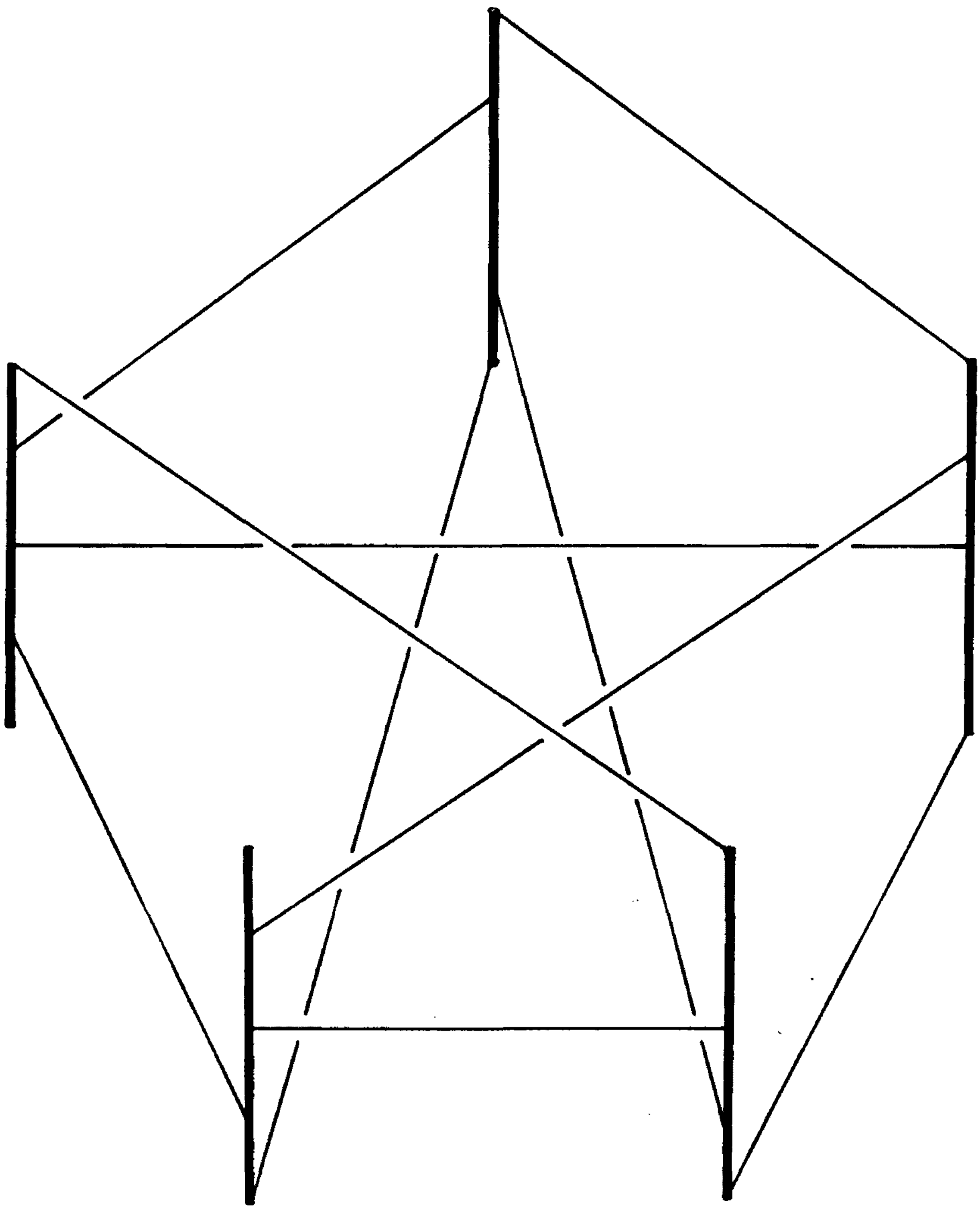


Figure 4.10

in $\Omega(LS^2)$ but it is in $\Omega(LS^3)$ as shown in figure 4.10.

Suppose $G\in\Omega(LS^n)$ with $n\geq4$. We show that $G\in\Omega(LS^{n-1})$, and this will complete the proof. Fix a line segment representation for G in R^n. We wish to project to R^{n-1} via an orthogonal projection. Such a projection will not destroy any intersections between line segments, but may create new intersections. We show, however, that almost all projections do not cause new intersections.

The space of all orthogonal projections of R^n to an (n-1)-dimensional subspace can be identified with the space of all (n-1)-dimensional subspaces which, in turn, by taking orthogonal complements, be identified with the set of all lines through the origin, better known as the (n-1)-dimensional manifold RP^{n-1}. Similarly, a projection which causes non-intersecting line segments to intersect after projection corresponds to a line through the origin which, when suitably translated, intersects both line segments. Now through any two points there is exactly one straight line. Hence the set of lines through the origin that translate to intersect both of a pair of line segments can be parameterized by 2 parameters. Thus the projections that cause non-intersecting line segments to intersect correspond to an (at most) two dimensional subset of RP^{n-1}. Since G is finite, the set of "forbidden" projections is contained in a finite union of 2-dimensional subsets of RP^{n-1}. Since $n-1\geq3>2$, almost all projections will not cause new intersections, and hence $G\in\Omega(LS^{n-1})$.∎

Are all graphs in $\Omega(LS^3)$? The answer is "no" as we shall see.

4.3.6 Theorem. $\Omega(LS^3)$ does not contain all graphs; in particular, it does not contain the graph $C_4\vee\beta K_6$.

Proof. Suppose $C_4\vee\beta K_6\in\Omega(LS^3)$. Now since C_4 is not an interval graph, the four line segments constituting C_4 are not collinear. Let L_1 and L_2 be line segments which are noncollinear and intersect each other,

representing two (adjacent) vertices of C_4. Let $\{x\}=L_1\cap L_2$. All vertices of C_4 are adjacent to all vertices of βK_6, thus all the line segments assigned to βK_6 intersect both L_1 and L_2. Now at most one of the "vertex-like" vertices of βK_6 has a line segment which contains the point x. Ignoring that particular "vertex-like" vertex and the 5 attached "edge-like" vertices, we have a line segment representation of βK_5 in which the line segments are constrained to lie in the plane spanned by L_1 and L_2, implying $\beta K_5\in\Omega(\mathbf{LS}^2)$, which is false. Hence $\Omega(\mathbf{LS}^3)$ does not contain all graphs.∎

4.3.7 Remark. Not every graph is in $\Omega(\mathbf{LS}^3)$, but perhaps every graph is in $\Omega(t\mathbf{LS}^3)$ for some t. To show that this is not the case we need only show that some bipartite graph is not in $\Omega(\mathbf{LS}^3)$. Unfortunately $C_4\vee\beta K_6$ is not bipartite. We therefore need a more complicated example.

4.3.8 Definition. Let G_1 and G_2 be bipartite graphs with $V(G_i)=X_i\cup Y_i$ be a partition of $V(G_i)$ into independent sets, for i=1,2. Let the **bipartite join**, denoted $G_1\vee_2G_2$, be the graph formed by taking the disjoint union of G_1 and G_2, and joining every vertex in X_i to every vertex in Y_{3-i}, for i=1,2.

Notice that the definition of bipartite join is ambiguous; it depends on the choice of the sets X_i and Y_i. In our application, however, one of the graphs will be $K_{3,3}$ and the ambiguity will disappear.

4.3.9 Definition. A surface (in $\mathbf{R}^3$) is called a **ruled** surface if it can be parameterized by $(s,t)\mapsto f(s)+tg(s)$, with $f,g:\mathbf{R}^1\to\mathbf{R}^3$. In other words, a surface is ruled if it is the union of a one-parameter family of lines. A **doubly ruled surface** is a surface which has two "essentially" different rulings, i.e. if $f_1(s)+tg_1(t) = f_2(s')+t'g_2(s')$, then $g_1(s)$ and $g_2(s')$ are linearly

independent.

There are relatively few doubly ruled surfaces:

4.3.10 Theorem [64,38]. The doubly ruled surfaces are:
the hyperbolic paraboloid, which is homeomorphic to a disc, and
the hyperboloid of one sheet, which is homeomorphic to an annulus.∎

4.3.11 Theorem [64,38]. Through three mutually skew lines in $\mathbf{R}^3$ there is exactly one doubly ruled surface, and every line which meets these three lines is contained in that surface.∎

4.3.12 Theorem. The bipartite graph $K_{3,3}v_2\beta K_5$ is not in $\Omega(\mathbf{LS}^3)$.

Proof. Suppose $K_{3,3}v_2\beta K_5 \in \Omega(\mathbf{LS}^3)$, and fix a line segment representation. Consider the 6 line segments representing $K_{3,3}$. If any two non-intersecting line segments are coplanar, it is easy to see that all 6 must be. It follows that the segments representing βK_5 are also in that plane, which is impossible.

Suppose instead that each pair of non-intersecting line segments from $K_{3,3}$ are skew. It follows that if we consider 3 non-intersecting segments, they determine a unique doubly ruled surface containing the other 3 line segments, as well as all the segments from βK_5. However, since every doubly ruled surface is homeomorphic to a subset of the plane, we would have (by that homeomorphism) a representation of βK_5 by curves in the plane, which is impossible.∎

4.3.13 Corollary. For every integer t there exists a graph which is not the intersection graph of sets which are each the union of up to t line segments in a real vector space.

Proof. By 4.3.5 it is enough to show that $\Omega(t\mathbf{LS}^3)$ does not contain all graphs. Let $P=\Omega(\mathbf{LS}^3)$. By 4.3.12, P does not contain all bipartite graphs, hence there exists, by 4.2.10, a graph with $P\#(G)>t$, thus G is not in $\Omega(t\mathbf{LS}^3)$.∎

4.4 Digression: Generalized Boxicity

We have shown how one can generalize the concept of interval graph to interval number, and then generalize further to P-number for arbitrary classes of graphs **P**. In her thesis, Cozzens [10] follows a similar route beginning with interval graphs, then boxicity and then generalizing boxicity to a parameter she denotes ϕ_P for intersection classes **P**. We call this parameter the "P-intersection number" and we give a theorem about its growth first proved by Kahn and Saks [41], but presented here with a shorter proof.

4.4.1 Definition. Let **P** be a class of graphs and let G be graph. The **P-intersection number** of G [resp. **P-union number**], denoted $P_\cap(G)$ [resp. $P_\cup(G)$], is the least positive integer t such that $G = H_1\cap\ldots\cap H_t$ [resp. $H_1\cup\ldots\cup H_t$] with $H_i\in P$.

4.4.2 Remark. Note that proposition 1.1.19 implies that if **P** denotes the class of interval graphs, we have $P_\cap(G)=b(G)$. It is not true, however, that $P_\cup(G)=i(G)$. For example if $G=K_{3,5}$ then $i(G)=2$, but it can easily be verified that $P_\cup(G)=3$. One can show however:

4.4.3 Proposition. If **P** is a class of graphs closed under disjoint union, then $P_\cup(G)\geq P\#(G)$ for all graphs, G.

Proof. Let $t=P_\cup(G)$. Let $G=H_1\cup\ldots\cup H_t$, and let $H=H_1+\ldots+H_t$. Since **P** is closed under disjoint union, $H\in P$. Let $V(G)=V(H_i)=\{v_1,\ldots,v_n\}$. Let $V(H)=\{v_i^s: 1\leq i\leq n,\ 1\leq s\leq t\}$, and $v_i^s\sim v_j^s$ if and only if $v_i\sim v_j$ in H_s. Let

$f:V(H)\to V(G)$ by $f(v_i^s)=v_i$. Clearly f is simplicial with multiplicity t, hence $P\#(G)\le t$.∎

We now study **P**-union and **P**-intersection numbers for monotone **P**.

4.4.4 Proposition [41]. Let **P** be monotone. The **P**-union number is well defined provided **P** contains K_2+nK_1 for all n. The **P**-intersection number is well defined provided **P** contains K_n-e, where e is an edge, for all n.∎

4.4.5 Definition. Recall that monotone classes of graphs **P** are ideals in the poset $(\mathbf{G},\le)$. Let $\bar{\mathbf{P}}$ denote the family of all complements of graphs in **P**:

$\bar{\mathbf{P}} = \{\bar{G}:\ G\in\mathbf{P}\}$.

Also, let $F(\mathbf{P})$ denote the minimal (with respect to $\le$) graphs not in **P**:

$F(\mathbf{P})=\{G$: G is not in **P**, but for all $H<G$, $H\in\mathbf{P}\}$.

Thus a graph G is not in **P** if and only if there exists $H\le G$ with $H\in F(\mathbf{P})$. To relate forbidden subgraphs of **P** and $\bar{\mathbf{P}}$ we note that $G\le H$ if and only if $\bar{G}\le\bar{H}$. This implies $G\in F(\mathbf{P})$ if and only if $\bar{G}\in F(\bar{\mathbf{P}})$.

4.4.6 Proposition. Let **P** be monotone. Then $\mathbf{P}_\cup(G) = \bar{\mathbf{P}}_\cap(\bar{G})$.

Proof. Let $\mathbf{P}_\cup(G)=t$, and let $G=H_1\cup\ldots\cup H_t$ with $H_i\in\mathbf{P}$. Then $\bar{G}=\bar{H}_1\cap\ldots\cap\bar{H}_t$. We claim that $\bar{H}_i\in\bar{\mathbf{P}}$. If $\bar{H}_i$ were not in $\bar{\mathbf{P}}$ then there exists $H\in F(\bar{\mathbf{P}})$ such that $H\le\bar{H}_i$ which implies $\bar{H}\le H_i$. But then $\bar{H}\in F(\mathbf{P})$ which implies H_i is not in **P** which is a contradiction. Thus $\bar{\mathbf{P}}_\cap(\bar{G})\le\mathbf{P}_\cup(G)$. The opposite inequality is similarly proved.∎

4.4.7 Theorem [41]. Let **P** be a monotone family of graphs which does not contain all graphs. Then the $\mathbf{P}_\cup$ and $\mathbf{P}_\cap$ parameters grow arbitrarily large.

Proof. Proposition 4.4.6 implies we need only prove this theorem for $\mathbf{P}_{\cup}$. Let $t>0$ be an integer, and G be a graph not in $\mathbf{P}$. By 0.5.3 we can find a graph G' such that every t coloring of the edges of G' contains an induced subgraph isomorphic to G in which all the edges have the same color.

Suppose $\mathbf{P}_{\cup}(G')\leq t$. Let $G'=H_1\cup\ldots\cup H_t$ with $H_i\in\mathbf{P}$. Color edge $xy\in E(G')$ with color i if $xy\in E(H_i)$. [In case xy is in $E(H_{i_1}),E(H_{i_2}),\ldots$ we can assign xy a color arbitrarily from $\{i_1,i_2,\ldots\}$.] This is a t coloring of $E(G')$. Hence there is an induced copy of G in G' in which all edges have the same color; say color 1. Let the vertices in G' which induce this monochromatic G be $v_1,\ldots,v_n$. If $v_i\sim v_j$ in G' with $1\leq i,j\leq n$ then $v_i\sim v_j$ in H_1. Conversely, if $v_i\sim v_j$ in H_1 we must have $v_i\sim v_j$ in G' since G' is the union of the H's. This implies there is an induced copy of G in H_1, i.e. $G\leq H_1\in\mathbf{P}$ which implies $G\in\mathbf{P}$, which is a contradiction. Thus $\mathbf{P}_{\cup}(G')>t$. ∎

4.4.8 Remark. The reason we have included this result is to compare it to 4.2.12. For monotone $\mathbf{P}$ we know $\mathbf{P}_{\cup}$ and $\mathbf{P}_{\cap}$ are unbounded if and only if $\mathbf{P}$ fails to contain some graph. Since $\mathbf{P}_{\cup}$ seems to resemble $\mathbf{P}\#$ one might expect the same behavior from $\mathbf{P}\#$. We know, however, that $\mathbf{P}\#$ grows unbounded if and only if $\mathbf{P}$ fails to contain some bipartite graph. The appearance of bipartite graphs seems somewhat surprising at this point. The "reason" bipartite graphs play this role is because edges have two endpoints. What happens when edges have more than two endpoints? We will examine that question in the next section.

We conclude this section with one more parameter which lies "between" $\mathbf{P}\#$ and $\mathbf{P}_{\cup}$:

4.4.9 Definition. Let $f:V(H)\rightarrow V(G)$ be a simplicial cover. We say that f is a **homomorphic cover** if $f^{-1}(x)$ is an independent set in H for all $v\in V(G)$.

Note that G is a homomorphic image of H in the usual graph theoretic sense.

4.4.10 Definition. Let **P** be a family of graphs and G be a graph. The **P-homomorphism number** of G, denoted $P_h\#(G)$, is the least positive integer t so that G can be covered by a graph H in **P** with a homomorphic cover of multiplicity t.

The following proposition explains the word "between" in the above remark. Its proof is trivial.

4.4.11 Proposition. If **P** is a class of graphs and G is a graph, then $P_h\#(G)\geq P\#(G)$ and in case **P** is closed under disjoint union, $P_h\#(G)\leq P_U(G)$.∎

We now show that the **P**-homomorphism number behaves similarly to the **P**-number.

4.4.12 Theorem. If **P** is a monotone class of graphs, then the **P**-homomorphism number is bounded if and only if **P** contains all bipartite graphs.

Proof. Suppose **P** contains all bipartite graphs. One readily checks that the simplicial cover constructed in the proof of 4.2.10 is a homomorphic cover, and so $P_h\#(G)\leq 2$ for all graphs G in this case.

Suppose **P** does not contain all bipartite graphs. Then the **P**# is unbounded, but since $P_h\#(G)\geq P\#(G)$ for all graphs G, the result

follows.∎

4.5 Irrepresentability by Multinerves

In this section we extend the results of sections 4.1 and 4.2 to both k-uniform hypergraphs and to nerves.

4.5.1 Definition. Let K and H be k-uniform hypergraphs. A function $f:V(K)\to V(H)$ is called a **simplicial map** provided:

(1) f is onto,

(2) if $e\in E(K)$ and $f|e$ is one-to-one, then $f(e)\in E(H)$, and

(3) if $e=\{v_1,\ldots,v_k\}\in E(H)$ then there exist $w_i\in f^{-1}(v_i)$ for $1\le i\le k$ such that $\{w_1,\ldots,w_k\}\in E(K)$.

Notice that when k=2 this definition agrees exactly with 4.2.2.

4.5.2 Definition. If **P** is a family of k-uniform hypergraphs and K is a k-uniform hypergraph, then the **P-number** of K, denoted **P**#(K), is the least positive integer t such that there exists $H\in$**P** and a simplicial map $f:V(H)\to V(K)$ of multiplicity t.

4.5.3 Theorem. If **P** is a monotone family of k-uniform hypergraphs then the **P**-number is bounded if and only if **P** contains all k-uniform k-partite hypergraphs.

Proof. Suppose **P** contains all k-partite k-uniform hypergraphs and let K be a k-uniform hypergraph. We show that **P**#(K)≤k. Let $V(K)=\{v_1,\ldots,v_n\}$. Let H be the k-partite k-uniform hypergraph defined as follows: Let $V(H)=\{v_i^s:1\le i\le n \text{ and } 1\le s\le k\}$. The edges of H are defined to be the sets

$$\{v_{i_1}^{\pi(1)},\ldots,v_{i_k}^{\pi(k)}\}$$

where π is a permutation of $\{1,\ldots,k\}$ and $\{v_{i_1},\ldots,v_{i_k}\}\in E(K)$. Clearly H is k-uniform. It is also k-partite since two vertices of H with the

same superscript cannot belong to the same edge. Hence $H\in\mathbf{P}$. Let $f:V(H)\to V(K)$ by $f(v_i^s)=v_i$. It is immediate that f is a simplicial map of multiplicity k. Thus $\mathbf{P}\#(K)\le k$.

Conversely, suppose K is k-uniform, k-partite and not in $\mathbf{P}$. Let K' be the k-uniform, k-partite hypergraph which, by 0.5.4 has the property that for every T coloring of the edges of K' there exists an induced copy of K in K' all of whose edges have the same color. Take $T=t^k$. We claim that $\mathbf{P}\#(K')>t$.

Suppose $\mathbf{P}\#(K')\le t$ and let $H\in\mathbf{P}$ such that $f:V(H)\to V(K')$ is simplicial with $\mathrm{mult}(f)=t$. Let $V(K')=V_1\cup\ldots\cup V_k$ be a k-partition of the vertices of K'. Let $\{u_i^s\}=V(H)$ with $f(u_i^p)=f(u_i^q)$ for all $1\le p,q\le t$. Let $e=\{w_1,\ldots,w_k\}\in E(K')$ with $w_i\in V_i$. Let $\{u_1^{s(1)},\ldots,u_k^{s(k)}\}\in E(H)$ such that $f(u_i^{s(i)})=w_i$. Color edge e with the k-tuple $(s(1),s(2),\ldots,s(k))$.

By the way K' is defined there exists a monochromatic induced copy of K in K'. We may assume by permuting the superscript of the u's, that the color is $(1,1,\ldots,1)$ without loss of generality. Let the vertices of the monochromatic K be $\{v_1,\ldots,v_n\}$, and let $\{u_i^1,\ldots,u_i^t\}=f^{-1}(v_i)$. We claim that $\{u_1^1,\ldots,u_n^1\}$ induces a copy of K in H. This is easy to see since $\{v_{i_1},\ldots,v_{i_k}\}\in E(K)$ if and only if $\{u_{i_1}^1,\ldots,u_{i_k}^1\}\in E(H)$. Thus $K\le H\in\mathbf{P}$ which implies $K\in\mathbf{P}$, a contradiction. Thus $\mathbf{P}\#(K')>t$.∎

4.5.4 Remark. Note that the proof of 4.5.3 is essentially the same as that of 4.2.10, only we changed all the 2's to k's. Theorem 4.5.3 is interesting because it clarifies the role of bipartite graphs in 4.2.10. We will also use it in studying "multi-nerves."

For the remainder of this section we consider k to be a fixed positive integer, and let S denote a family of sets.

4.5.5 Definition. Let $\Omega^*(S)$ be the family of (k+1)-uniform hypergraphs for which there exists an injection $f:V(H)\to S$ so that $f(v_0)\cap\ldots\cap f(v_k)\neq\emptyset$ if and only if $\{v_0,\ldots,v_k\}\in E(H)$. Let $\Omega'(S)$ be the subclass of $\Omega^*(S)$ of all hypergraphs, H, whose representing function f has the additional property that for any k+2 vertices $\{v_0,\ldots,v_{k+1}\}$ we have $f(v_0)\cap\ldots\cap f(v_{k+1})=\emptyset$.

Note that we now focus on (k+1)-uniform hypergraphs because they can be most readily related to k-dimensional simplicial complexes.

4.5.6 Proposition. Let $P^*=\Omega^*(S)$ and $P'=\Omega'(S)$. If H is a (k+1)-uniform hypergraph, then $P^*\#(H)=t$ [resp. $P'\#(H)=t$] if and only if t is the smallest integer for which $H\in\Omega^*(tS)$ [resp. $H\in\Omega'(tS)$].

Proof. Analogous to 4.2.4.∎

4.5.7 Definition. Let

$N^*(S)=\{sk_kK:\ K\in N(S)\}$, and

$N'(S)=\{K\in N(S):\ \dim(K)\leq k\}$.

4.5.8 Definition. A simplicial complex is called **k-equimaximal** if and only if each of its maximal edges has k+1 vertices. If K is a k-dimensional simplicial complex we define a new complex μK as follows: Let the maximal edges of K be $\{e_1,\ldots,e_p\}$. Let $q_i=k+1-|e_i|$ and let $f_i=\{w_{i,1},\ldots,w_{i,q_i}\}$ be new vertices. (In case $q_i=0$ then $f_i=\emptyset$.) The vertices of μK are $V(K)\cup f_1\cup\ldots\cup f_p$ and the maximal edges are $\{e_1\cup f_1,e_2\cup f_2,\ldots,e_p\cup f_p\}$. See figure 4.11.

4.5.9 Definition. Let $\mathcal{S}$ be the class of all k-dimensional simplicial complexes, let $\mathcal{M}$ be the class of all k-equimaximal simplicial complexes, and let $\mathcal{U}$ be the class of all (k+1)-uniform hypergraphs. There is a natural bijection between $\mathcal{U}$ and $\mathcal{M}$ given by the following

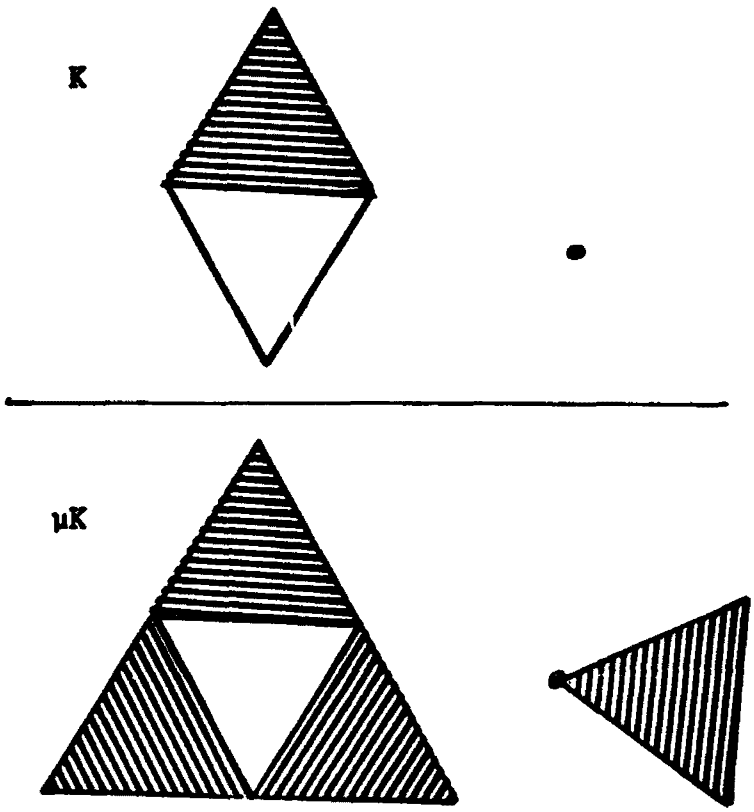


Figure 4.11

pair of functions:

Define $u:\mathcal{M}\rightarrow\mathcal{U}$ as follows: If K is a k-equimaximal simplicial complex, then $V(uK)=V(K)$ and $E(uK)=\{e\in E(K):|e|=k+1\}$.

Define $\sigma:\mathcal{U}\rightarrow\mathcal{M}$ as follows: If H is a (k+1)-uniform hypergraph let $V(\sigma H)=V(H)$ and $E(\sigma H)=\{e:\emptyset\neq e\subset e' \text{ for some } e'\in E(K)\}$.

Clearly for $K\in\mathcal{M}$ and $H\in\mathcal{U}$ we see that $\sigma uK=K$ and $u\sigma H=H$. The functions σ, μ and u and the classes $\mathcal{S}$, $\mathcal{M}$ and $\mathcal{U}$ are shown in Figure 4.12.

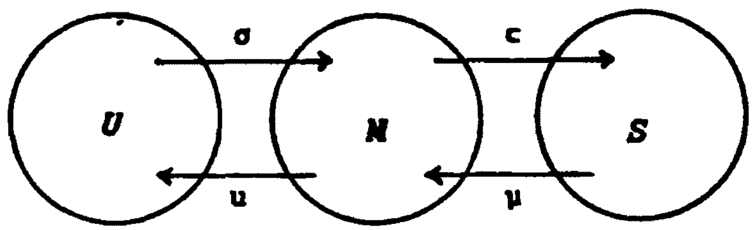


Figure 4.12

4.5.10 Proposition. Let $H\in\mathcal{U}$. Then $H\in\Omega^*(S)$ if and only if $\sigma H\in N^*(S)$ and $H\in\Omega'(S)$ if and only if $\sigma H\in N'(S)$.

Proof. One easily checks that the same function f can be used to show membership in either class in either case.∎

4.5.11 Lemma. $\Omega^*(S)=\mathcal{U}$ if and only if $N^*(S)=\mathcal{S}$, and $\Omega'(S)=\mathcal{U}$ if and only if $N'(S)=\mathcal{S}$.

Proof. Suppose $\Omega^*(S)=\mathcal{U}$. Let $K\in\mathcal{S}$. We know $u\mu K\in\mathcal{U}=\Omega^*(S)$ thus $\sigma u\mu K=\mu K\in N^*(S)$. Since $K\leq\mu K$ and $N^*(S)$ is easily seen to be monotone, we have $K\in N^*(S)$. Thus $N^*(S)=\mathcal{S}$.

Conversely, if $N^*(S)=\mathcal{S}$ and $H\in\mathcal{U}$ we know $\sigma H\in\mathcal{S}=N^*(S)$, thus $H\in\Omega^*(S)$. Therefore $\Omega^*(S)=\mathcal{U}$.

The proof for Ω' and N' is analogous.∎

4.5.12 Theorem. Let $P=N^*(S)$. There exists an integer t with the property that $N^*(tS)=\mathcal{S}$ if and only if P contains all (k+1)-partite k-dimensional simplicial complexes. [Respectively for N'.]

Proof. Let $P=N^*(S)$ and $Q=\Omega^*(S)$. Suppose P contains all (k+1)-partite

k-dimensional simplicial complexes. It follows that if H is any (k+1)-uniform (k+1)-partite hypergraph that $\sigma H \in \Omega^*(S)=\mathbf{Q}$. Thus $\mathbf{Q}$ contains all (k+1)-partite (k+1)-uniform hypergraphs and so $\Omega^*((k+1)S)=\mathcal{U}$. Thus $\mathbf{N}^*((k+1)S)=\mathcal{S}$.

Conversely, suppose K is a k-dimensional (k+1)-partite simplicial complex not in $\mathbf{P}$. Since $K \le \mu K$ we know μK is not in $\mathbf{P}$. Thus $u\mu K$ is not in $\mathbf{Q}$. Since $u\mu K$ is (k+1)-partite we know that for all t we have $\Omega^*(tS)\neq\mathcal{U}$, Thus for all t we also have $\mathbf{N}^*(tS)\neq\mathcal{S}$. [An analogous argument holds for $\mathbf{N}'$.]∎

4.5.13 Corollary. The following statements are equivalent:

(1) There exists an integer t such that $\mathbf{N}(tS)$ contains all k-dimensional simplicial complexes.

(2) There exists an integer t such that every k-dimensional simplicial complex is the k-skeleton of an element of $\mathbf{N}(tS)$.

(3) $\mathbf{N}(S)$ contains all (k+1)-partite k-dimensional simplicial complexes.

Proof. This follows immediately from theorem 4.5.12. The equivalence of (1) and (3) follows by considering $\mathbf{N}'$ and the equivalence of (2) and (3) by considering $\mathbf{N}^*$.∎

4.5.14 Remark. It would be natural at this point to apply these results to nerves of convex sets in some Euclidean space. Since $\mathbf{N}(K^{2k+1})$ contains all k-dimensional complexes, we would naturally want to show that for any integer t the class $\mathbf{N}(tK^{2k})$ does not contain all k-dimensional simplicial complexes. To do so we need only provide an example of a (k+1)-partite k-dimensional simplicial complex which is not in $\mathbf{N}(K^{2k})$. In case k=1 we saw that the graph βK_5 is the appropriate example. Unfortunately no such examples are known for $k\ge 2$. We do however have the following eminently reasonable conjecture due to Wegner [77]:

4.5.15 Conjecture. If K is a simplicial complex which does not embed in $\mathbf{R}^{2k}$ then βK is not in $N(K^{2k})$.□

It is known that there are k-dimensional simplicial complexes which do not embed in $\mathbf{R}^{2k}$ [76,78]. If this conjecture is true, then since the barycentric subdivision of a k-dimensional simplicial complex is (k+1)-partite, we would have the necessary example. Indeed, Wegner's conjecture seems very plausible. The topological details seem tedious but should not be terribly difficult. It even should be possible to prove the following somewhat stronger version:

4.5.16 Conjecture. If K is a simplicial complex which does not embed in $\mathbf{R}^{2k}$ then βK is not the nerve of sets each of which is a cell in $\mathbf{R}^{2k}$ with the additional property that the non-empty intersection of any of these cells is again a cell.□

5. INTERSECTION REPRESENTATIONS FOR PLANAR GRAPHS

In this section we consider multiple intersection representations for planar graphs. Perhaps the most impressive result in this field is the following theorem due to Thurston:

Theorem [66]. Every planar graph is the intersection graph of circles in the plane. In addition, one may assume that intersecting circles are tangent.∎

Our main results are that planar graphs are intersection graphs of sets each of which is the union of

- three real intervals, or
- two boxes in $\mathbf{R}^2$, or
- two line segments in $\mathbf{R}^2$.

The results concerning multiple interval representations of planar graphs are joint work with D.B. West [62] and are included here for completeness.

We begin, in section 5.1, with a general discussion of planar graphs. Sections 5.2 and 5.3 concern multiple interval representations of planar graphs. Section 5.4 concerns multiple interval representations for graphs with given genus. Section 5.5 discusses two dimensional box representations and section 5.6 discusses two dimensional line segment representations of planar graphs.

5.1 Facts About Planar Graphs

Since planar graphs will be central to our discussion, we include here several theorems concerning planar graphs to which we shall refer in subsequent sections of this chapter.

5.1.1 Definition. A **planar graph** is a graph which has an embedding in the plane. By a **plane graph** we mean a planar graph together with one of its embeddings in the plane. Although planar graphs may have several essentially different embeddings, a plane graph is a planar graph with one of those embeddings considered to be "standard". One cannot speak of a vertex on the unbounded face of a planar graph, since the embedding is not specified, but one can discuss such notions for plane graphs.

5.1.2 Definition. An **outerplanar graph** is a plane graph all of whose vertices are on the unbounded face. A vertex of a plane graph is called an **outervertex** in case it is on the unbounded face. (Thus an outerplanar graph is a plane graph all of whose vertices are outer.) A vertex of a plane graph is an **innervertex** if it is not an outervertex.

The **outer induced subgraph** of a plane graph G is the induced subgraph on G's outervertices and is denoted oG. If v is an outervertex of G its **outerneighbors** are those outervertices to which it is adjacent. Its other neighbors are called **innerneighbors**.

5.1.3 Definition. An edge joining two outervertices is called **external** if it lies on the unbounded face; otherwise it is called a **chord**.

5.1.4 Definition. The **dual** of a plane graph G is a graph G* for which there is a vertex for each face of G and these vertices are adjacent if and only if the corresponding faces share an edge. The **weak dual** of G, denoted wG, is the induced subgraph of G* formed by deleting the vertex corresponding to the unbounded face.

5.1.5 Theorem [15]. Let G be a plane graph. G is an outerplanar graph if and only if wG is a forest. wG is a tree if and only if G is a 2-connected outerplanar graph.∎

5.1.6 Definition. A **leaf face** of an outerplanar graph G is a face corresponding to a vertex of degree at most one in wG.

5.1.7 Definition. A **block** of a graph is a maximal 2-connected induced subgraph of G. A **cut vertex** of a graph G is a vertex whose removal results in a graph having more connected components than G.

The **block graph** of G, denoted bG, is a graph with a vertex for each block of G with two vertices being adjacent if and only if the corresponding blocks share a cut vertex.

5.1.8 Theorem. If G is a graph, then bG is a forest and G is connected if and only if bG is a tree.∎

5.1.9 Definition. A **leaf block** of a graph G is a block corresponding to a leaf (vertex of degree at most one) of bG.∎

5.1.10 Definition. Let G be a graph. The genus of G, denoted $\gamma(G)$, is the least positive integer g for which G embeds on an (orientable) surface of genus g.

The next three results can be found in [1].

5.1.11 Theorem. $\gamma(K_n)=\lceil(n-3)(n-4)/12\rceil$ and $\gamma(K_{n,m})=\lceil\frac{1}{4}(n-2)(m-2)\rceil$.∎

5.1.12 Theorem. The Euler characteristic χ and the genus g of an (orientable) surface are related by $\chi=2-2g$. If $\gamma(G)=g$ then $|E(G)|\leq3(|V(G)|-\chi)$. In case G is triangle-free, we can replace "3" by "2".∎

5.1.13 Theorem. Let g be a non-negative integer and $n=\lfloor \frac{1}{2}(7+\sqrt{(1+48g)})\rfloor$. Then K_n is the largest complete graph with $\gamma(K_n)=g$. Moreover, if $\gamma(G)=g$, then G is n-colorable.∎

5.1.14 Theorem. Let $a(g)=\max\{T(G): \gamma(G)=g\}$. (See §2.2.5 for the definition of arboricity of G, denoted T(G).) $a(0)=3$. For $g>0$, we have

$$\lceil\sqrt{(3g)}\rceil \le a(g) \le 2+\lceil\sqrt{(3g)}\rceil.$$

Proof. Let G be a graph and let H be an induced subgraph such that $n=|V(H)|$, $m=|E(H)|$, and $T(G)=\lceil m/(n-1)\rceil$. (See Theorem 2.2.7.)

Suppose G is planar. This implies H is planar, hence $m\le 3n-6$. Thus $T(G)\le\lceil(3n-6)/(n-1)\rceil \le \lceil(3(n-1)/(n-1)) - 3/(n-1)\rceil \le 3$. Thus $a(0)\le 3$. Since the arboricity of K_5 less an edge is 3 and K_5-e is planar we see $a(0)=3$.

Suppose $\gamma(G)=1$. Since $H\le G$, either $\gamma(H)=0$, implying $T(H)\le 3$ and thus $T(G)\le 3$, or else $\gamma(H)=1$. In the latter case $m\le 3(n-0)$, so $\lceil m/(n-1)\rceil \le \lceil 3n/(n-1)\rceil = \lceil 3 + 3/(n-1)\rceil = 4$. Thus $a(1)\le 4$. Since $\gamma(K_7)=1$ and $T(K_7)=\lceil\frac{1}{2}7\rceil=4$, the theorem is verified for g=1.

We now suppose g>1 and proceed by induction. We consider two cases: $n\le$ or $n>\frac{1}{2}(7+\sqrt{(1+48g)})$. In the first case since H is a subgraph of K_n, we know $T(G) = T(H) \le T(K_n) = \lceil\frac{1}{2}n\rceil \le \lceil\frac{1}{2}(7+\sqrt{(1+48g)})\rceil \le \lceil\frac{1}{2}(8+\sqrt{(48g)})\rceil = 2+\lceil\sqrt{(3g)}\rceil$.

Suppose $n>\frac{1}{2}(7+\sqrt{(1+48g)})$. If $\gamma(H)<g$ then by induction $T(G)=T(H)\le a(g-1)$. So we need only consider $\gamma(H)=g$. In this case $m\le 3(n-(2-2g))=3n+6g-6$. Thus:

$$\begin{aligned} m/(n-1) &\le (3n+6g-6)/(n-1) = \\ &= 3(n-1)/(n-1) + (6g-3)/(n-1) = \\ &= 3 + (6g-3)/(n-1). \end{aligned}$$

Since $n > \frac{1}{2}(7+\sqrt{(1+48g)})$ we have

$$
\begin{aligned}
m/(n-1) &\le 3 + (6g-3)/[\tfrac{1}{2}(5+\sqrt{(1+48g)})] = \\
&= 3 + (12g-6)/[5+\sqrt{(1+48g)}] = \\
&= 3 + [(12g-6)(5-\sqrt{(1+48g)})]/[25-(1+48g)] = \\
&= 3 + [6(2g-1)(5-\sqrt{(1+48g)})]/[-24(2g-1)] \\
&= 3 + \tfrac{1}{4}(\sqrt{(1+48g)} - 5) = \\
&= \tfrac{1}{4}(7+\sqrt{(1+48g)}) \le \\
&\le 2 + \lceil \sqrt{(3g)} \rceil
\end{aligned}
$$

as required.

Now if $p=\lfloor \frac{1}{2}(7+\sqrt{(1+48g)}) \rfloor$, then $\gamma(K_p)=g$. Also, $T(K_p) = \lceil \frac{1}{4} \lfloor \frac{1}{2}(7+\sqrt{(1+48g)}) \rfloor \rceil \ge \lceil \frac{1}{4}(6+\sqrt{(1+48g)}) \rceil \ge \frac{1}{4}\sqrt{(48g)} = \sqrt{(3g)}$. Thus $a(g)$ satisfies the stated bounds.∎

5.1.15 Lemma (Pie-Slice). Consider a plane graph consisting of a chordless cycle $v_1{\sim}v_2{\sim}\ldots{\sim}v_n{\sim}v_1$ and two vertices u and w surrounded by this cycle. Let the neighbors of u be $\{v_{i_1}, v_{i_2}, \ldots, v_{i_p}\}$ with $i_1<\ldots<i_p$. Let P_j denote the path $v_{i_j}{\sim}v_{i_j+1}{\sim}\ldots{\sim}v_{i_{j+1}}$ (with subscript addition modulo n) for $1 \le j \le p$. All of w's neighbors are contained in one of the paths P_j.

Proof. This is obvious from figure 5.1. We also give a formal proof. Suppose w had neighbors x and x' with x in path P_j but not in $P_{j'}$, and x' in path $P_{j'}$ but not in P_j. Recall that v_{i_j} and $v_{i_{j'}}$ are the first vertices in these paths and both are adjacent to u. Choose a point z external to the cycle and join it by edges to x, x', v_{i_j} and $v_{i_{j'}}$. Since no vertices of our original plane graph are outside the v-cycle, this gives a new plane graph. See figure 5.2. However, also observe that this plane graph is homeomorphic to K_5 (the "vertices" of the K_5 being z, x, x', v_{i_j} and $v_{i_{j'}}$) which is impossible.∎

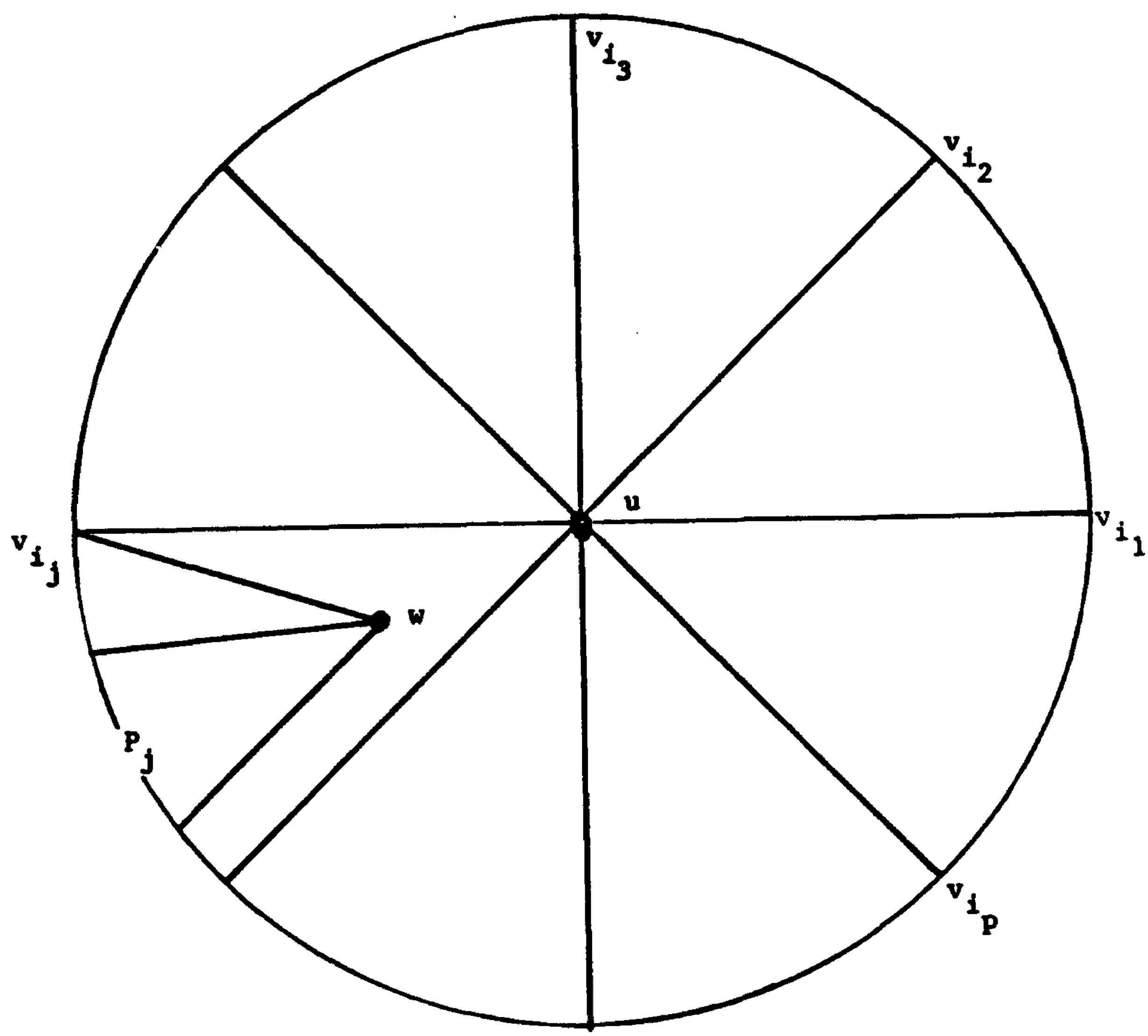


Figure 5.1

5.2 Three Intervals Per Vertex: Necessity

It is our aim in the next two sections to show that the maximum value of the interval number of a planar graph is three. In this section we show that $i(K_{2,9}^{+})=i^{+}(K_{2,9})>2$.

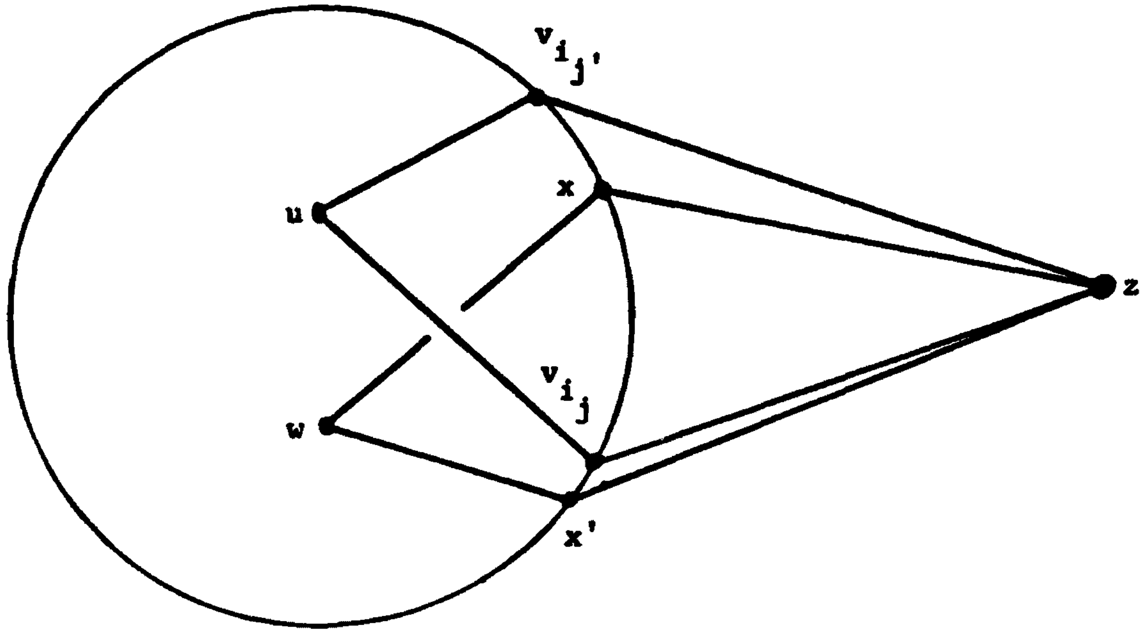


Figure 5.2

5.2.1 Definition. Let G be a graph and let f be a t-interval representation of G. Let $v \in V(G)$ and $f(v)=I_1 \cup I_2 \cup \ldots \cup I_s$ where the I_j are pairwise disjoint real intervals and $s \le t$. We assume $I_j=[a_j,b_j]$ and $a_1<b_1<\ldots<a_s<b_s$.

A **broken end** for vertex v is an endpoint of one of its intervals I_j which is contained in no other intervals in the representation of G. The **effective number of broken ends** for vertex v, denoted $\beta(v)$, is the number of a_i's and b_i's which are broken ends, plus 2(t-s). We add 2(t-s) to the actual number of broken ends because one can always augment the representation by adding t-s additional intervals for v in an unused portion of the real line, adding 2(t-s) more broken ends. We denote the total number of broken ends in the representation by $\beta^* = \Sigma\beta(v)$, where the sum is over all $v \in V(G)$.

5.2.2 Definition. Let G be a graph and let v and w be vertices of G with $v \sim w$. Let f be a t-interval representation for G. We know $f(v) \cap f(w) \neq \emptyset$. It is only required that one interval for v intersect one interval for w; any additional intersections are redundant. Define $\rho(v,w)$ to be the number of **extra** times intervals for v intersect intervals for w. [In case v and w are not adjacent, put $\rho(v,w)=0$.] Define the **redundancy** of vertex v by $\rho(v)=\Sigma\rho(v,w)$ with the sum over all $w \in V(G)$. Put $\rho^*=\Sigma\rho(v)$.

5.2.3 Proposition. ρ^* and β^* are even.

Proof. Real broken ends come in pairs; every right broken end "faces" a left broken end (see figure 5.3) except the very leftmost and the very

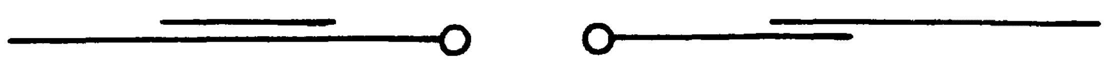


Figure 5.3

rightmost, which also form a pair. Thus β^* is even.

Concerning ρ^* we know $\rho^*=\Sigma_v\rho(v)=\Sigma_v\Sigma_w\rho(v,w)$. Since $\rho(v,v)=0$ and $\rho(v,w)=\rho(w,v)$ it is immediate that ρ^* is even.∎

5.2.4 Proposition. The total number of nonempty intersections between pairs of intervals in a t-interval representation of a graph G is $|E(G)| + \frac{1}{2}\rho^*$.

Proof. Every vertex v has intervals which make k(v) intersections with other intervals. By definition, $k(v)=d(v)+\rho(v)$. Thus $\Sigma k(v)=2|E(G)|+\rho^*$. Since $\Sigma k(v)$ counts every intersection twice, the result follows.∎

5.2.5 Proposition. If G is a graph on n vertices with a depth-2 t-interval representation f, then $|E(G)| = nt - \frac{1}{2}(\beta^* + \rho^*)$.

Proof. We assume that f(v) is the union of t disjoint intervals. [If not we can always add additional intervals in an unused portion of the real line. This affects neither β^* nor ρ^*.] Let $I_1, I_2, \ldots, I_{nt}$ be the intervals in f labeled in increasing order of left endpoints.

Count the total number of intersections among these nt intervals. In a depth-2 representation, each I_j meets at most one interval with lower subscript; otherwise the left endpoint of I_j is contained in two other intervals, violating depth-2. Thus each interval I_j meets 0 or 1 intervals of lower subscript depending precisely on whether or not its left endpoint is broken. Since every intersection of intervals pairs one interval with a second of lower subscript, the number of intersections between pairs of intervals is $nt - \frac{1}{2}\beta^*$. However, in 5.2.4 we saw that this same quantity equals $|E(G)| + \frac{1}{2}\rho^*$, thus $|E(G)| = nt - \frac{1}{2}(\rho^* + \beta^*)$.∎

5.2.6 Proposition. Given a displayed (see §2.1.14) t-interval representation of a graph G and a vertex v we have

$$\rho(v)+\beta(v)+d(v) > t.$$

In case the representation need not be displayed we have

$$\rho(v)+\beta(v)+d(v) \geq t.$$

Proof. In case $\rho(v)+\beta(v)>t$ we are done, so assume $\rho(v)+\beta(v) \leq t$. Recall that the number of intersections between intervals assigned to v and other intervals is $\rho(v)+d(v)$. Let $j=t-\rho(v)-d(v)$. If $j<0$ then we are done, so we assume $j \geq 0$. Thus there are at least j intervals for v that intersect no other intervals (or do not appear in the representation). Thus $\beta(v) \geq 2j$. Since $\rho(v)=t-d(v)-j$ we get

$$\beta(v)+\rho(v) \geq 2j+(t-d(v)-j) = j+t-d(v), \text{ or}$$

$$\beta(v)+\rho(v)+d(v) \geq t+j \geq t,$$

giving the latter part of the proposition. In case the representation is displayed we are also done unless $j=t-\rho(v)-d(v)=0$ and $\beta(v)=0$. However, in this case there are exactly t intersections between intervals for v and other intervals. No interval for v is unused (or fails to intersect another interval) for then $\beta(v)\geq 2$. Thus v is assigned t disjoint intervals, each of which intersects at least (and hence at most) one interval. Thus all t intervals must be wholly contained in the interval which they intersect. It follows that vertex v is not displayed, contrary to supposition.∎

5.2.7 Theorem. $i(K^+_{2,9})>2$. Since $K^+_{2,9}$ is planar, the maximum value of the interval number for planar graphs is at least three.

Proof. $i(K^+_{2,9})=i^+(K_{2,9})$ by §2.1.16. Suppose $i^+(K_{2,9})=2$ and form a displayed 2-interval representation for $K_{2,9}$. Since $K_{2,9}$ is triangle-free, such a representation is necessarily depth-2. Now each of $K_{2,9}$'s nine vertices of degree two must satisfy $\rho(v)+\beta(v)+d(v)>2$, or $\rho(v)+\beta(v)>0$. Thus $\rho^*+\beta^*\geq 9$. By §5.2.5 we know

$$|E(K_{2,9})| = 2|V(K_{2,9})| - \tfrac{1}{2}(\rho^*+\beta^*)$$

or

$$18 = 2(11) - \tfrac{1}{2}(\rho^*+\beta^*) \leq 22 - \tfrac{1}{2}9 = 17.5.$$

This contradiction implies $i^+(K_{2,9})>2$, or $i(K^+_{2,9})>2$.∎

5.3 Three Intervals Per Vertex: Sufficiency

We show in this section that if G is a planar graph then $i(G)\leq 3$. We derive this by induction, however our induction hypothesis is rather involved. The main idea in the proof is that plane graphs have a displayed 3-interval representation in which outervertices are assigned at most two intervals per vertex. Then as additional vertices are added to the graph, outervertices become innervertices and the additional interval now permitted allows flexibility in forming necessary edges.

Numerous technical assumptions must be placed on the representation of the graph in order for this idea to work. We begin by presenting these assumptions. Note that we assume in our approach that the plane graph G is connected; this results in no loss of generality since if the interval number of each component of a graph is at most three, clearly the interval number of the graph is at most three.

5.3.1 Definition. Let G be a connected plane graph. A **root** for G is an outervertex which is not a cut vertex of the outer induced subgraph oG. A **rooted** plane graph is a pair (G,z_0) where G is a connected plane graph and z_0 is a root.

The rooting of a connected plane graph G induces a rooting of the tree boG; the root of boG corresponds to the block of oG containing the root vertex.

5.3.2 Definition. Let (G,z_0) be a rooted plane graph, and let H be a block of oG. The **entry vertex** of H, denoted z(H) is the first vertex of H encountered in any path in G from z_0 to H. In case $z_0 \in V(H)$, clearly $z(H)=z_0$. Otherwise, z(H) is the cut vertex of oG belonging to H closest to z_0. Edges of H containing z(H) are called **entry edges**. See figure 5.4.

5.3.3 Definition. Let (G,z_0) be a rooted plane graph and let H be a block of oG. The **fan** of H, denoted F(H), is an induced subgraph of H on those vertices which are in faces containing z(H). See figure 5.5. Notice that the fan of H consists of all the entry edges of H and a path containing the outerneighbors of z(H) in H. However, not all vertices in F(H) are necessarily adjacent to z(H).

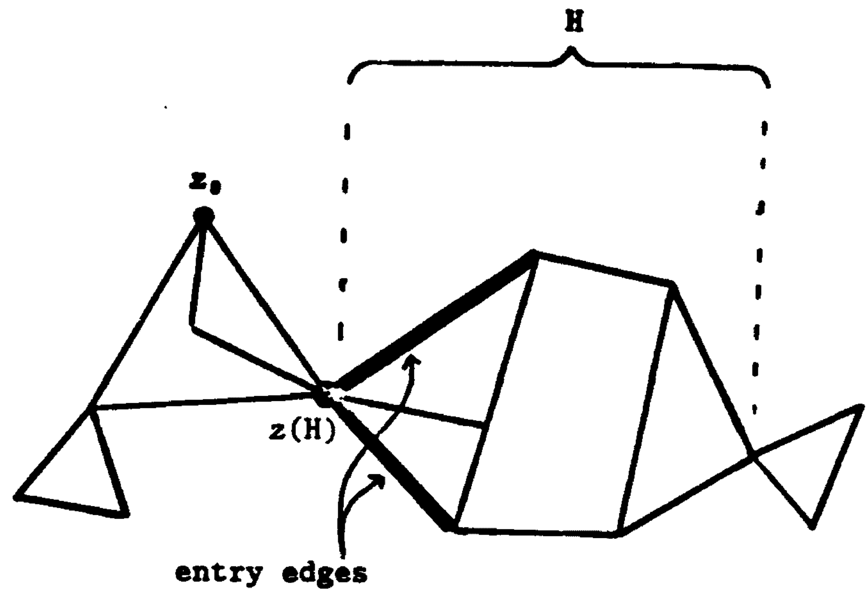


Figure 5.4

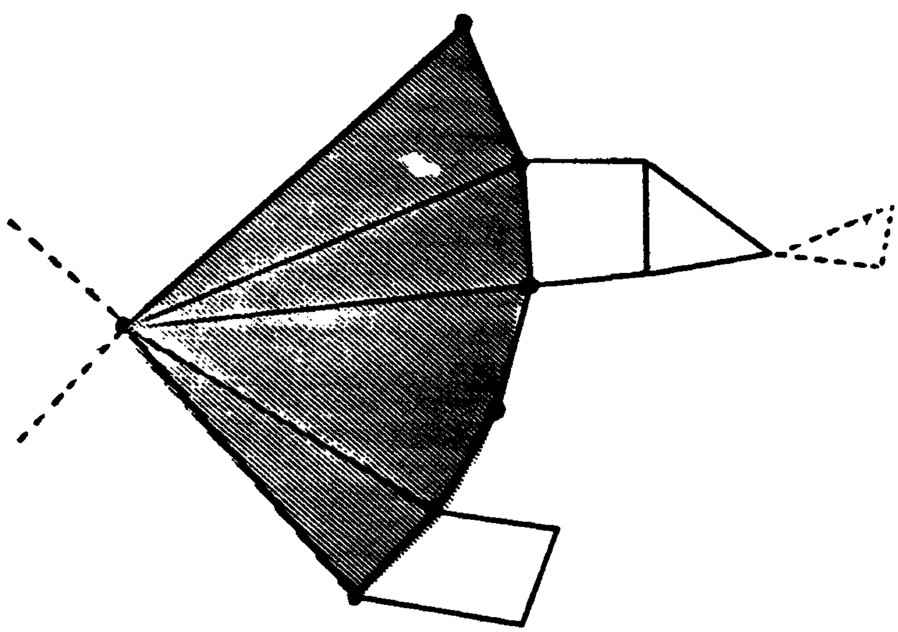


Figure 5.5

5.3.4 Definition. Let G be a connected plane graph, let xy be a chord of G, and let u,z be two other vertices. Observe that G-x-y is not

connected. If u and z are in the same connected component of G-x-y then u is said to be on the **z-side** of xy. Otherwise u is said to be on the **anti-z-side** of xy.

We now direct our attention to properties our representation will have.

5.3.5 Definition. Let G be a plane graph and let $f:V(G)\rightarrow 2^{\mathbf{R}}$ be an intersection representation for G. We say that f is a **multiple interval representation** for G is f(v) is always a finite union of intervals for all $v\in V(G)$. We say that f is a **displayed 2/3-interval representation** (or 2/3-representation for short) for G provided:

(1) for each outervertex v, f(v) is the union of (up to) two intervals,

(2) for each innervertex v, f(v) is the union of (up to) three intervals, and

(3) for each vertex v, f(v) contains a nonempty open set (known as the **displayed portion** for v) which does not intersect any f(w) for $w\neq v$.

5.3.6 Definition. Let G be a plane graph and let f be a multiple interval representation for G. Let $xy\in E(G)$. We say xy is **displayed** in f provided $f(x)\cap f(y)$ contains a nonempty open set (known as the **displayed portion** for xy) which does not intersect f(w) for all $w\in V(G)-\{x,y\}$.

5.3.7 Definition. Let (G,z_*) be a rooted plane graph, let f be a 2/3-representation, let xy be an edge of oG which is not an entry edge, let H be the block of oG containing xy, and let b be an endpoint of an interval in f(x). We say that b is a **reusable endpoint** for xy provided either:

(1) b is a broken end (i.e. not contained in any other interval),

or

(2) b belongs to f(u) for exactly one other vertex $u \in V(H)$ ($u \neq x$, $u \neq y$) where ux is an external edge of G and u is on the anti-z-side of xy.

In the latter case we say u **covers** b.

An **assignment** of reusable endpoints is a function $\xi : E' \to \mathbf{R}$ where $E' \subset E(oG)$ and if $\xi(xy)=b$ then b is a reusable endpoint for xy. (Thus $b \in f(x) \cup f(y)$.) Such an assignment is called **proper** if:

(1) E' = {those edges of oG not displayed in f},

(2) ξ is one-to-one,

(3) no outervertex u covers more than one $\xi(xy)$ for $xy \in E'$, and

(4) if $xy \in E'$ and xy is external then $\xi(xy)$ is a broken end.

5.3.8 Definition. Let (G,z_0) be a rooted plane graph and let f be a multiple interval representation for G. We say that f is a **P-special interval representation** provided:

(1) f is a displayed 2/3-interval representation for G,

(2) every entry edge of oG is displayed, and

(3) there is a proper assignment, ξ, of reusable endpoints.

5.3.9 Theorem [62]. If (G,z_0) is a rooted plane graph then G has a P-special interval representation.

Proof. We proceed by induction. The theorem is trivial in case $V(G)=\{z_0\}$. Assume $|V(G)| \geq 2$ and that the theorem is true for all graphs with fewer vertices. Let H be a leaf block of oG. We assume z_0 is not a vertex of H unless H is the only block of oG. Let $z=z(H)$. Either $H=F(H)$ or $H>F(H)$. In either case we will delete one or more vertices of H and form a P-special interval representation for the resulting induced subgraph of G. We then show how to augment this representation to one for G.

Case I: H=F(H)

This case is illustrated in figure 5.6. Label the vertices of H

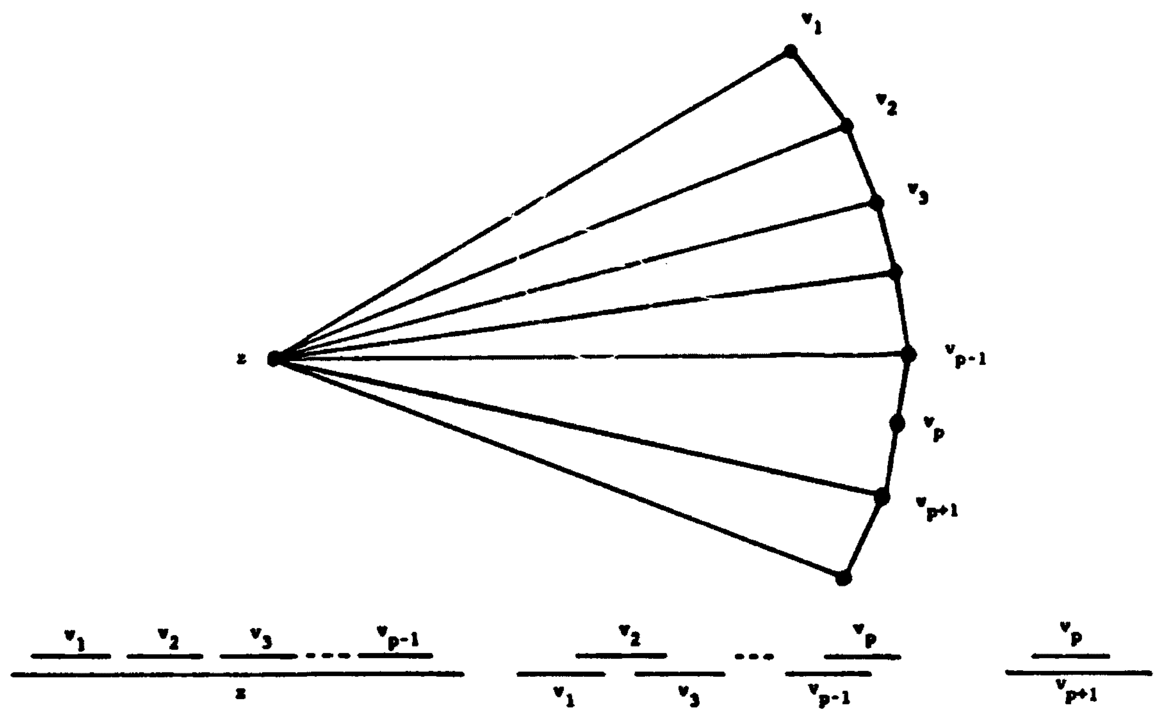


Figure 5.6

as $z, v_1, v_2, \ldots, v_k$ in order on the external face. Let $v_{k+1}=z$. In case k=1 we see that H consists of a single edge. Form a P-special interval representation f for $G-v_1$. We extend f to G by assigning two intervals to v_1: one in an unused portion of the line and the other in the displayed portion of z. Observe that, so extended, f forms a P-special interval representation for G.

In case k>1 we observe that $d_H(v_1)=d_H(v_k)=2$ and $d_H(v_i)=2$ or 3 for $1<i<k$. Let $p=\min\{i: 2\le i\le k, d_H(v_i)=2\}$. Let $V=\{v_1,\ldots,v_p\}$. Form a P-special interval representation f for G-V. We modify f to form a P-special representation for G as follows (see figure 5.6):

For each $v \in V$ we assign two interval. For $v_1, \ldots, v_{p-1}$ place one interval each in a disjoint fashion in the displayed portion for z. Place one interval for v_p in the displayed portion for v_{p+1}. Also, for the vertices in V, place a second displayed interval for each in an overlapping consecutive fashion as shown in figure 5.6. Observe that all edges zv_i and v_iv_{i+1} for $1 \leq i \leq p$ are displayed. We must also represent edges between V and innervertices of G. Let w be an arbitrary innervertex of G adjacent to one or more vertices in V. Since w is an outervertex of G-V we know that f(w) is the union of two intervals. We may therefore assign a third interval for w in G's representation. Notice that w cannot be adjacent to two nonconsecutive vertices in V; were w adjacent to v_i and v_j with $i<j-1$ then one of the edges wv_i or wv_j would cross wv_{i+1}, violating planarity. Thus w is adjacent to at most two vertices in V and those vertices must be consecutive. If w is adjacent to one v_i, place a third interval for w in the displayed portion of v_i. In case w is adjacent to v_i and v_{i+1}, place w's third interval in the displayed portion for edge v_iv_{i+1}. Repeating for all such vertices w, we claim we arrive at a P-special interval representation for G. Observe that chords of G-V that are not chords of G may now not have reusable endpoints, but they do not need them. The representation of vertices and edges of oG not in H are not affected by our emendations. One easily verifies that G now has a P-special representation.

Case II: H>F(H)

Choose a leaf face C of H which does not contain z; since H>F(H) this is always possible. There is a unique chord of H between this leaf face and the rest of H; call this chord xy. (See figure 5.7.) We consider two cases: (a) C is a triangle, or (b) C has 4 or more vertices.

Subcase IIa: C is a triangle

The vertices of C are {x,y,v} where $d_H(v)=2$. Form a P-special interval representation for G-v. Now edge xy of G-v may either be displayed or have a reusable endpoint. In case xy is displayed we assign two intervals to v: one in the displayed portion of xy and a second in an unused portion of the real line. (See figure 5.7.) Observe that xy

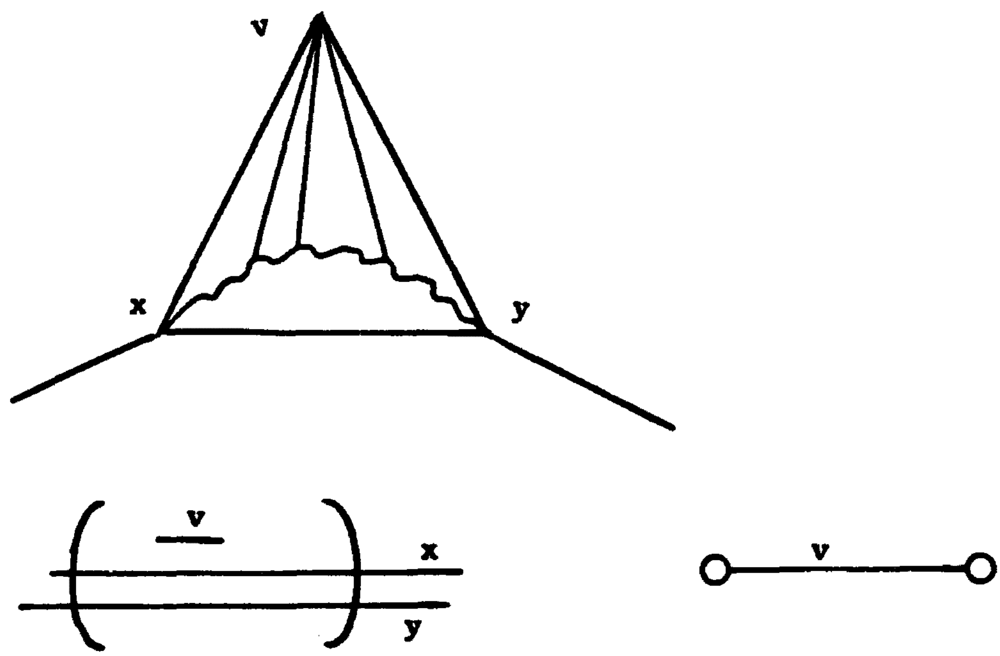


Figure 5.7

is still displayed but xv and yv are not. We assign the left endpoint of the second interval as a reusable endpoint for xv and the right endpoint to yv. Since v is on the anti-z-side of xy and since both endpoints are broken this extension of the assignment of reusable endpoints for G is still proper. Now v may have innerneighbors w in G which were outer in G-v. We therefore assign a third interval to each such w in v's displayed portion, completing the P-special interval representation for G.

We must still consider the case when xy is not displayed but rather assigned a reusable endpoint b which we may suppose is an endpoint of an interval for x. Suppose b is covered by u. Now xu is an external edge in G-v but since u, like v, is on the anti-z-side of xy we see that u becomes an innervertex in G. We therefore split the interval for u covering b in two exposing b. (See figure 5.8.) We now place one interval for v containing b in its interior as shown and a second interval in the displayed portion of y. See figure 5.8.

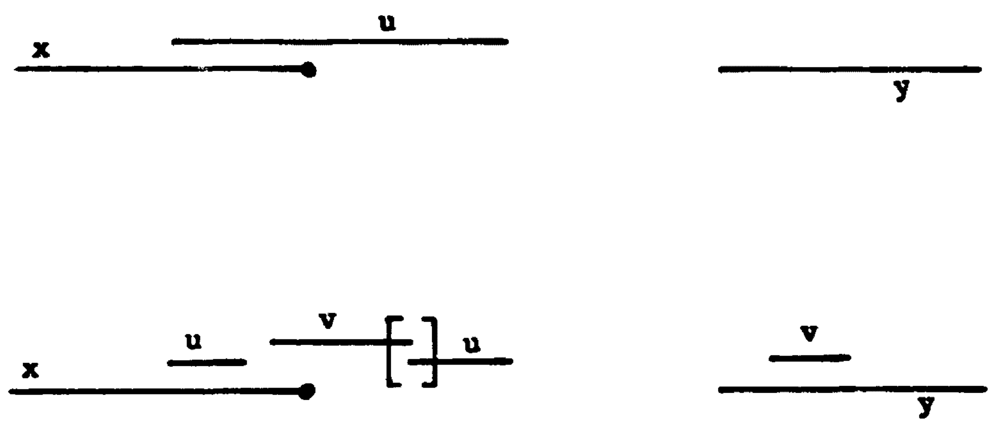


Figure 5.8

Observe that edges xv and yv are displayed. Edge xy still has b as its reusable endpoint, only now v is its cover. Finally we consider the innerneighbors of v. If v~u we may overlap intervals for u and v as show by the square brackets in the figure. Since the other innerneighbors were outer in G-v, we may assign a third interval for each in the displayed portion for v. This gives a P-special interval representation for G. In case b was a broken end, simply ignore all mention of u in the above argument.

Subcase IIb: C is a chordless cycle with at least four vertices

Let the vertices of C be (in cyclic order): $x, v_1, v_2, \ldots, v_k, v_{k+1}=y$, where xy is the chord and all other edges are external. Let f be a P-special interval representation of $G-\{v_1,v_2\}$. To obtain a P-special interval representation for G proceed as follows:

(1) Assign v_1 an interval in the displayed portion for x and assign v_2 an interval in the displayed potion for v_3.

(2) This step depends on how xy is represented in f:

(2a) Edge xy is displayed or its reusable endpoint is a broken end: Place overlapping displayed intervals for v_1 and v_2 in an unused portion of the real line. Assign intervals for the remaining innerneighbors of v_1 and v_2 in the displayed portions for v_1, v_2, or v_1v_2 depending on which of $\{v_1,v_2\}$ the innervertex neighbors.

(2b) Edge xy's reusable endpoint is $b\epsilon f(x)$: The covering vertex u is outer in $G-\{v_1,v_2\}$, but inner in G sin ce C is a leaf face without chords. Hence there is now an extra interval available for u; we split the interval for u to expose b. Place overlapping intervals for v_1 and v_2 in the gap between b and the new broken end for u, with the interval for v_1 covering b. See figure 5.9. If either v_1 or v_2 is adjacent to u, extend the appropriate interval to create the optional overlap indicated by square brackets in figure 5.9. Assign intervals for the remaining innerneighbors of v_1 and v_2 in the displayed portions for v_1, v_2, or v_1v_2 depending on which of $\{v_1,v_2\}$ the innervertex neighbors.

(2c) Edge xy's reusable endpoint is $b\epsilon f(y)$: If k=2 we interchange the roles of x and y and proceed as in step

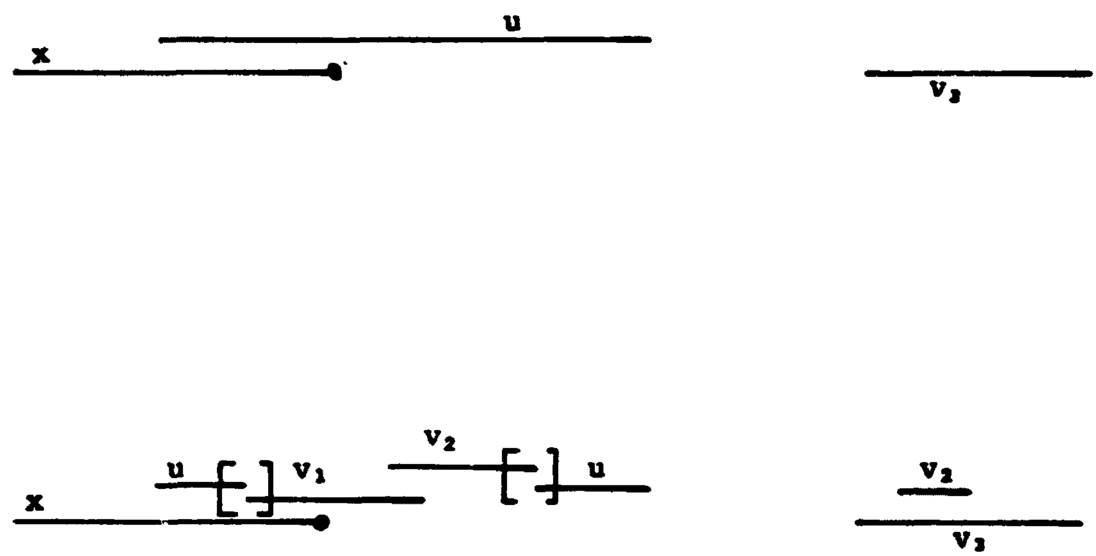


Figure 5.9

(2b). Suppose $k>2$. If $u=v_k$ we may proceed as in step (2a) with v_k continuing to cover b. If $u \neq v_k$ then it must be an innervertex in G. If u is not adjacent to both v_1 and v_2 then split the interval for u in order to expose b. Intervals for v_1 and v_2 are place as in figure 5.10 depending on which one (if either) is adjacent to u. Assign intervals for the remaining innerneighbors of v_1 and v_2 in the displayed portions for v_1, v_2, or v_1v_2 depending on which of $\{v_1,v_2\}$ the innervertex neighbors.

In case u is adjacent to both v_1 and v_2 we have a problem. In this special case let $x'=y$, $y'=x$ and $v_i'=v_{k-i+1}$ (we are reversing the roles of x and y). We now start over from step (1). No problem arises unless, again, we arrive at this case with edge $x'y'$ having reusable endpoint $b' \in f(y')=f(x)$ and b' is covered by innervertex u' with u' adjacent to both v_1' and v_2', i.e., u' is adjacent to x, v_k

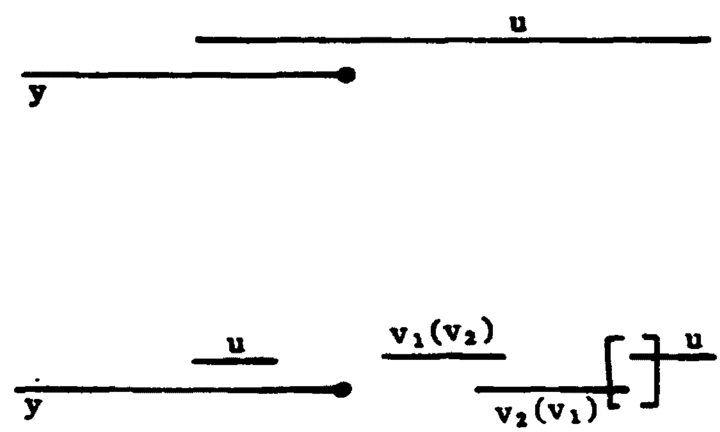


Figure 5.10

and v_{k-1}. We claim this is impossible. We know that $y \sim u \sim v_1$ and $x \sim u' \sim v_k$ with $v_1 \neq v_k$, and both u and u' internal to cycle C. The pie-slice lemma implies u=u', but then edge uy is not an external edge of $G-\{v_1,v_2\}$ as required by 5.3.7.

It is now routine to verify that we have constructed a P-special interval representation for G. Vertices v_1 and v_2 and external edges xv_1, v_1v_2 and v_2v_3 are all displayed. Edge xy is displayed or assigned a reusable endpoint as it was in f.∎

5.3.10 Corollary. If G is a planar graph, then $i(G) \leq 3$ and this bound is best possible.∎

5.3.11 Corollary. If G is an outerplanar graph, then $i(G) \leq 2$ and this bound is best possible.∎

5.3.12 Theorem. If G is a planar graph and $r \geq 3$, then $i_r(G) \leq 3$ and this bound is best possible.

Proof. Since G is planar it contains no K_5, hence if r≥4 we know $i_r(G)=i(G)$. In case r=3 we observe that in the proof of 5.3.9 that the P-special interval representation has depth-3.∎

5.3.13 Theorem. If G is planar, then $i_2(G)\le 4$ and this bound is best possible.

Proof. We saw in §5.1.14 that planar graphs G satisfy $T(G)\le 3$. Thus by 2.2.6 $i_2(G)\le 4$. To show that 4 is the best possible bound, consider the planar graph G shown in figure 5.11. Observe that G has 22 vertices and 60 edges with 13 vertices of degree 3. Suppose $\overset{+}{i_2}(G)\le 3$, i.e. G has a displayed 3-interval depth-2 representation. By 5.2.6, each vertex v of degree 3 satisfies $\rho(v)+\beta(v)>0$, hence $\rho^*+\beta^*\ge 13$. By 5.2.5 we have

$$|E(G)| = nt-\tfrac{1}{2}(\beta^*+\rho^*) \le 3(22)-\tfrac{1}{2}(13) = 59.5.$$

But $|E(G)|=60$, hence $\overset{+}{i_2}(G)\ge 4$, thus $i_2(G^+)\ge 4$. Since G^+ is planar, the proof is complete.∎

5.4 Maximum Interval Number of Graphs with Given Genus

In this section we extend the results of sections 5.2 and 5.3 by considering the maximum value of the interval number of graphs with fixed genus.

5.4.1 Theorem. If $\gamma(G)=g$ then $i(G)\le 3+\lceil\sqrt{(3g)}\rceil$.

Proof. By 5.3.10 this is correct when $\gamma(G)=0$. If $\gamma(G)>0$ then by 5.1.14 $T(G)\le 2+\lceil\sqrt{(3g)}\rceil$. Hence by 2.2.6 $i(G)\le i_2(G)\le 3+\lceil\sqrt{(3g)}\rceil$.∎

5.4.2 Theorem. Let $i_2max(g) = \sup\{i_2(G): \gamma(G)=g\}$. For g>0 we have

$$-1+\lceil\sqrt{(3g)}\rceil \le i_2max(g) \le 3+\lceil\sqrt{(3g)}\rceil.$$

Proof. The upper bound follows from the proof of 5.4.1 and the lower

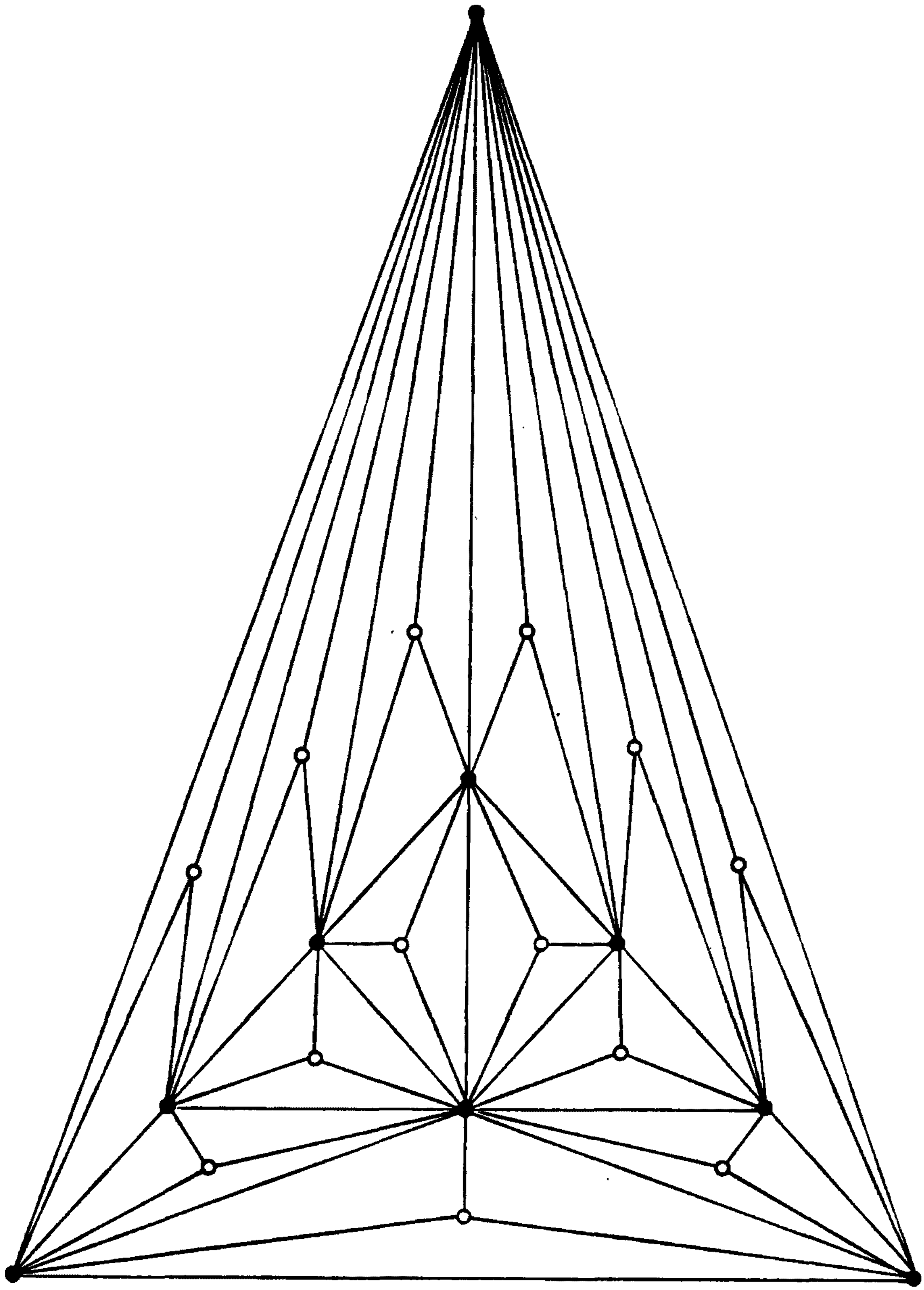


Figure 5.11

bound follows from 5.1.14 and 2.2.9.∎

5.4.3 Theorem. Let $\mathrm{imax}(g) = \max\{i(G): \gamma(G)=g\}$. Then we have the following:

$$\sqrt{g} \le \mathrm{imax}(g) \le 3+\lceil\sqrt{(3g)}\rceil.$$

Proof. The upper bound follows from 5.4.1. For the lower bound let $r=\lfloor 2+2\sqrt{g}\rfloor$ and let $G=K_{r,r}$. By 5.1.11 $\gamma(G) = \lceil \frac{1}{4}(r-2)^2\rceil \le \lceil \frac{1}{4}[(2+2\sqrt{g})-2]^2\rceil = g$. Since imax is clearly non-decreasing, $i(G)\le \mathrm{imax}(g)$. Now $i(G)$ is given by

$$\begin{aligned} i(G) &= i(K_{r,r}) = \\ &= \lceil (r^2+1)/(2r)\rceil \ge \\ &\ge \tfrac{1}{2}r = \\ &= \tfrac{1}{2}\lfloor 2+2\sqrt{g}\rfloor \ge \\ &\ge \sqrt{g}. \end{aligned}$$

This completes the proof.∎

5.5 Multiple Box Representations of Planar Graphs

In this section we show that every planar graph has an intersection representation by sets each of which is the union of two boxes in the plane. (Recall from §1.1.16 that a box in $\mathbf{R}^2$ is a closed rectangle with sides parallel to the coordinate axes.) In other words, if G is planar, then $G\in\Omega(2\mathbf{B}^2)$. This result is best possible since K(2,2,2) is planar, but b(K(2,2,2))>2 implying K(2,2,2) is not in $\Omega(\mathbf{B}^2)$.

Our approach is similar to the proof that $i(G)\le 3$ for planar G. We will assume that outervertices are assigned only one box each and that external edges have a special representation.

5.5.1 Definition. Let G be a graph. A **planar multiple box representation** for G is a function $f:V(G)\to 2^{\mathbf{R}^2}$ satisfying:

(1) f is an intersection representation for G, and

(2) for all $v\in V(G)$, f(v) is the union of finitely many boxes in $\mathbf{R}^2$.

5.5.2 Definition. Let f be a planar multiple box representation for a graph G. A vertex v of G is **displayed** in this representation provided there is an open disc centered at a point on the boundary of f(v) which does not intersect f(w) for any vertex $w\neq v$.

Let xy be an edge of G. Observe that $f(x)\cap f(y)$ is the union of boxes in an obvious manner. We say edge xy is **displayed** if there is an open disc centered at a corner of $f(x)\cap f(y)$ which is divided into four 90° sectors: one which meets f(x) only, one which meets f(y) only, one which meets f(x) and f(y) only, and one which meets no f(w) for any vertex w. See figure 5.12. The edge is called **doubly displayed** if it

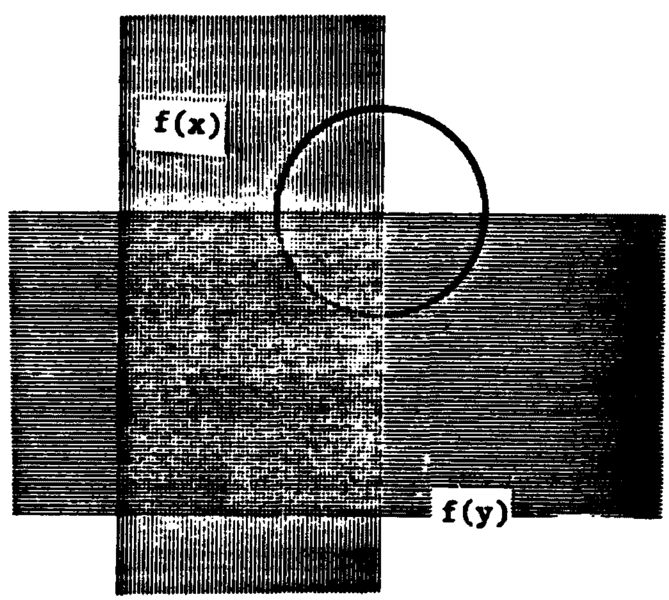


Figure 5.12

has two such discs which are disjoint.

5.5.3 Definition. Let f be a planar multiple box representation for a plane graph G. We say f is a **P-special** planar box representation for G provided:

(1) if v is an outervertex of G, then f(v) is a (single) box,

(2) if v is an innervertex of G, then f(v) is the union of (at most) two boxes,

(3) all vertices are displayed in f,

(4) all external edges are displayed in f, and

(5) if xy is an external edge and also an isthmus, then xy is doubly displayed.

We prove that all planar graphs are in $\Omega(2\mathbf{B}^2)$ by induction on the number of vertices. Indeed, we prove a stronger statement: Every planar graph has a P-special planar box representation. We need one additional concept for the proof:

5.5.4 Definition. Let B_0, B_1, B_2, ..., B_t be planar boxes with $t \geq 3$. These boxes form a **staircase arrangement framed by** B_0 and B_1 if they are arranged as in figure 5.13. The key property of this configuration is that each B_i, with $2 \leq i \leq t$ has B_0- and B_1-**clearance** by which we mean we can extend B_i to the left [resp. downward] to meet B_1 [resp. B_0] without forming any new intersections. This is because the lower right hand corner of B_{i-1} is above and to the left of the upper left hand corner of B_{i+1} for $3 \leq i \leq t-1$.

Naturally the orientation of this arrangement is not important and we may have a mirror image of this configuration.

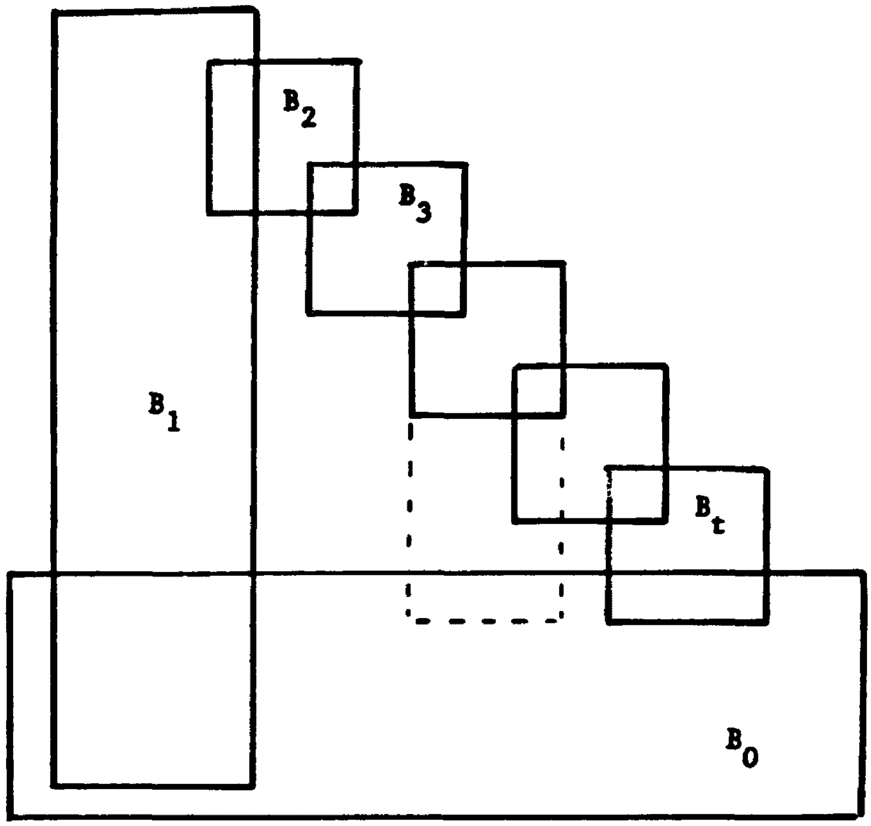


Figure 5.13

5.5.5 Theorem. If G is a plane graph, then G has a P-special planar box representation.

Proof. Let G be a plane graph. The result is obvious if G is a single vertex, so we assume that the theorem is true for all plane graphs with fewer than $|V(G)|$ vertices. We may assume G is connected since if it were not, each of its components would possess a P-special planar box representation.

Suppose oG has a vertex v of degree 1. Form a P-special planar box representation f for G-v. Let w be v's neighbor in G. Since w is an outervertex it is displayed in f. We can add a box for v to this

representation by having it meet w's box in its displayed portion (see figure 5.14). Notice that both v and w are now displayed and that edge

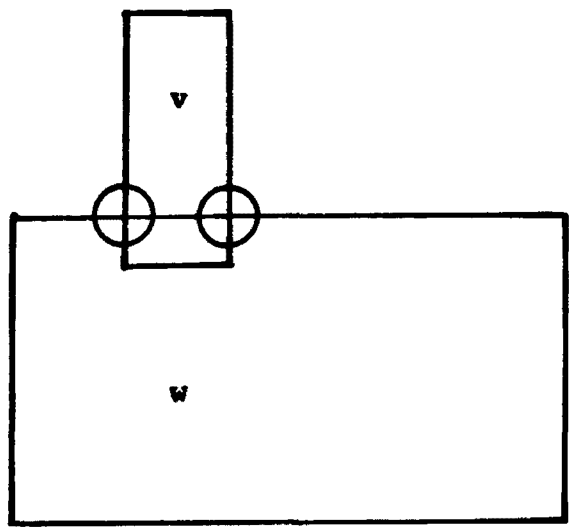


Figure 5.14

vw, an external isthmus, is doubly displayed.

(Notice that the above paragraph proves that the boxicity of a tree is at most two since every tree has a vertex of degree one whose removal gives a tree with fewer vertices.)

Suppose oG has no vertices of degree 1. As in the proof of 5.3.9 we can find a face C of oG in which all but two of the vertices have degree 2. Let the vertices of C be v_0, v_1, ..., v_k with $d_{oG}(v_i)=2$ for $i\geq 2$. Now C is a face of oG and may bound more vertices. Let H denote the plane graph induced on those innervertices of G bounded by C. Let G' be the induced subgraph of G formed by deleting $\{v_2,\ldots,v_k\}\cup V(H)$ from V(H). Form a P-special planar box representation f for G'+H. (Note that G'+H need not be an induced subgraph of G, since some vertices of H may be adjacent to v_0 or v_1.) Observe that v_0v_1 is an external edge of G'+H and is therefore displayed; and in case v_0v_1 is also external in G,

then it is an external isthmus of G'+H and therefore doubly displayed.

We will use one displayed portion of v_0v_1 in reconstructing the cycle C. Some outervertices of H may be adjacent to vertices of C. However, we may assign a second box to each to form these edges.

We consider two cases: k=2, i.e. C is a triangle, and k>2. Although one could consider the k=2 case as a special instance of the more general setting, we believe this separation into separate cases to be more enlightening.

Case I: k=2 (C is a triangle).

Our first step in extending f is to assign a box for v_2 contained in the open disc guaranteed by v_0v_1's being displayed. See figure 5.15. We place this box so that it contains in its interior the corner of

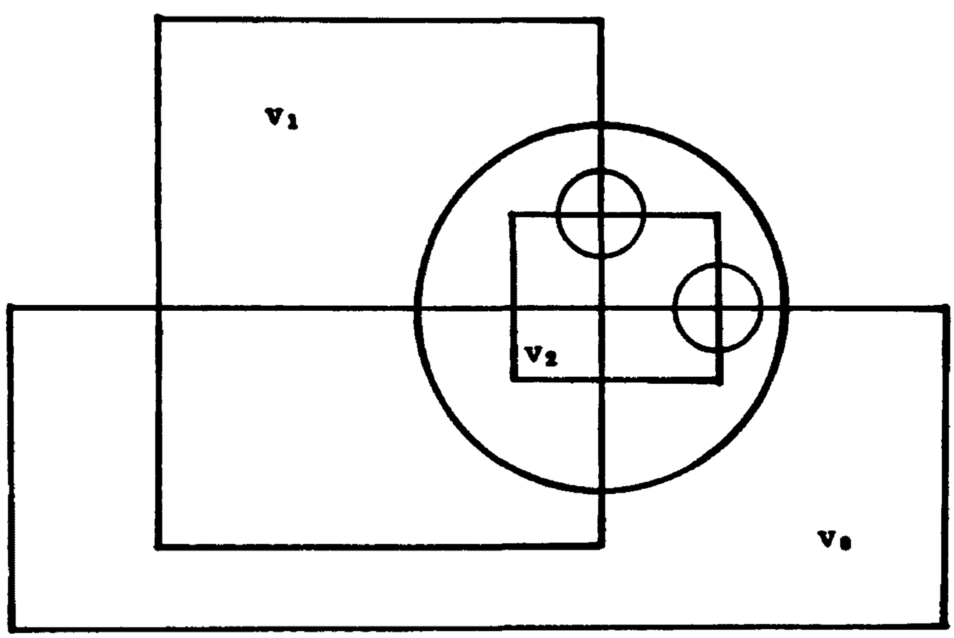


Figure 5.15

$f(v_0) \cap f(v_1)$ that is the center of this disc. Notice that edges v_0v_2 and v_1v_2 are displayed. If edge v_0v_1 needs to be displayed in G, it was previously doubly displayed and no problem arises.

Now outervertices of H may be adjacent to one or more of $\{v_0,v_1,v_2\}$. Since all vertices of H are inner in G we may assign to each an additional box. Notice that each of $f(v_i)-(f(v_j)\cup f(v_k))$, $(f(v_i)\cap f(v_j))-f(v_k)$, and $f(v_i)\cap f(v_j)\cap f(v_k)$ with $\{i,j,k\}=\{1,2,3\}$ contain open sets which do not meet any other boxes. Thus if $w\in V(H)$ is adjacent to any of v_1, v_2, v_3 we can place a second box in one of the aforementioned open sets to form the appropriate adjacencies. One now checks that we have constructed a P-special planar box representation for G.

Case II: $k>2$.

In this case we form the edges for C by constructing a staircase arrangement framed by $f(v_0)$ and $f(v_1)$ with new boxes for $v_2,\ldots,v_k$ forming the "stair" portion. This is done within the displayed disc for edge v_0v_1. See figure 5.16. Notice that all edges v_iv_{i+1} with $1\le i\le k$ and v_kv_0 are displayed. If edge v_0v_1 needs to be displayed in G then v_0v_1 was doubly displayed in G' and we still have a displayed "site" not yet used. We now show how, using the extra box available for each outervertex of H, to form edges between vertices in H and vertices in C. Suppose there are n such vertices in H. We show, in a recursive manner, how a second box for each of these n vertices can be placed into the staircase structure to form the appropriate edges. In case $n=0$ we are done. In case $n>0$ we proceed as follows:

• Some vertex u in H is adjacent to exactly one vertex in C: In this case we assign u's second box completely contained in the interior of the box for its neighbor in C meeting no other boxes. Repeat for all

such vertices.

• Some vertex u in H is adjacent to exactly two consecutive vertices of C: In this case we assign u's second box completely contained in the interior of the intersection of the boxes of its consecutive neighbors in C, meeting no other boxes. Repeat for all such vertices. All remaining vertices in H meet two non-adjacent vertices of C.

• Some vertex u in H meets two non-adjacent vertices in C: In this final case we suppose u meets $\{v_{i_1}, v_{i_2}, \ldots, v_{i_p}\}$ with $1<i_1<i_2<\ldots<i_p$. (u may be adjacent to v_0 and/or v_1 as well.) Please refer to figure 5.16. We place a thin box for u intersecting v_{i_1} and extending downward: if $u \sim v_0$ extend u's box down until it just penetrates the box for v_0. Otherwise the bottom side of u's box should be below the box for v_{i_p-1} but above the box for v_{i_p+1}. This thin box should be to the right to the box for v_{i_1-1} and to the left of the box for v_{i_1+1}.

Next extend the boxes for $v_{i_2}, \ldots, v_{i_p}$ to the left until they just penetrate into the box for u. Now if $u \sim v_1$ extend u's box to the left until it just penetrates into the box for v_1. One checks that u's box intersects exactly those boxes for vertices in C that it is supposed to intersect. Further, by the "pie-slice lemma" (§5.1.15) we can consider each "slice" of C separately for the remaining vertices in H. Notice that each "slice" gives rise to a staircase arrangement: For example $u, v_{i_j}, v_{i_j+1}, \ldots, v_{i_{j+1}}$ is a staircase framed by u and $v_{i_{j+1}}$. Also, if $v_1 \sim u$, then $v_1, v_2, \ldots, v_{i_1}, u$ is a staircase, otherwise $v_1, v_2, \ldots, v_{i_1}, u, v_{i_p}, v_{i_p+1}, \ldots, v_k, v_0$ is a staircase. In every case, further vertices of H (which are adjacent to two non-consecutive vertices in C) can be represented by adding a second box into the appropriate staircase structure. This may result in more and more staircases, but each time we add a vertex, the pie-slice lemma assures

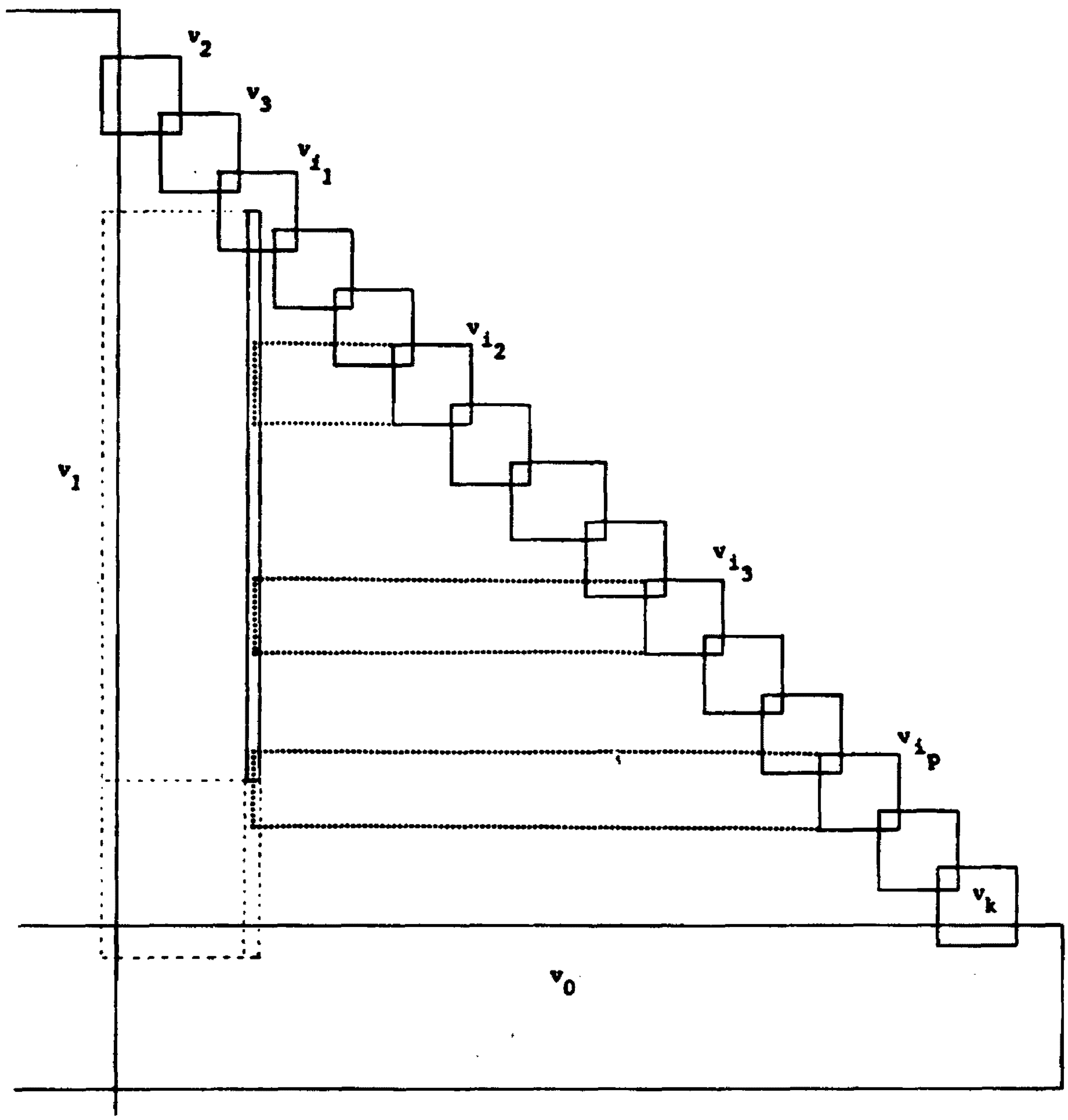


Figure 5.16

us that we need only insert into and modify only one of the staircases. In the end we have a P-special planar box representation for G.∎

5.5.6 Corollary. If G is an outerplanar graph, then $b(G) \le 2$.∎

5.5.7 Remark. We saw in the proof that the boxicity of a tree is at most two. Since the arboricity of a planar graph is at most three, we have an easy proof that every planar graph is in $\Omega(3\mathbf{B}^2)$.

For higher dimensions we know all planar graphs are in $\Omega(2\mathbf{B}^d)$ for all $d \ge 2$. However, we posit:

5.5.8 Conjecture. If G is planar, then $b(G) \le 3$.□

5.6 Multiple Line Segment Representations of Planar Graphs

In this section we show that every planar graph is the intersection graph of sets each of which is the union of two line segments in $\mathbf{R}^2$. In other words, if G is a planar graph, then $G \in \Omega(2\mathbf{LS}^2)$. As in previous proofs we proceed by induction with a somewhat elaborate induction hypothesis.

5.6.1 Definition. Let G be a graph. A **planar multiple line segment representation** for G is a function $f: V(G) \to 2^{\mathbf{R}^2}$ satisfying:

(1) f is an intersection representation for G, and

(2) for all $v \in V(G)$, $f(v)$ is the union of finitely many closed line segments in $\mathbf{R}^2$.

5.6.2 Definition. Let f be a planar multiple line segment representation for a graph G. A vertex v of G is **displayed** in this representation provided there is an open disc centered at a point of $f(v)$ which meets no $f(w)$ for any $w \ne v$.

Let v_1v_2 be an edge of G. We say that this edge is **displayed** in this representation if there exist distinct points $x_i \in f(v_i)$, $i=1,2$, and

a positive real number ε so that the ε-neighborhood N of the line segment joining x_1 and x_2 satisfies:

(1) $N \cap f(v_i)$ is an open line segment, i=1,2, and

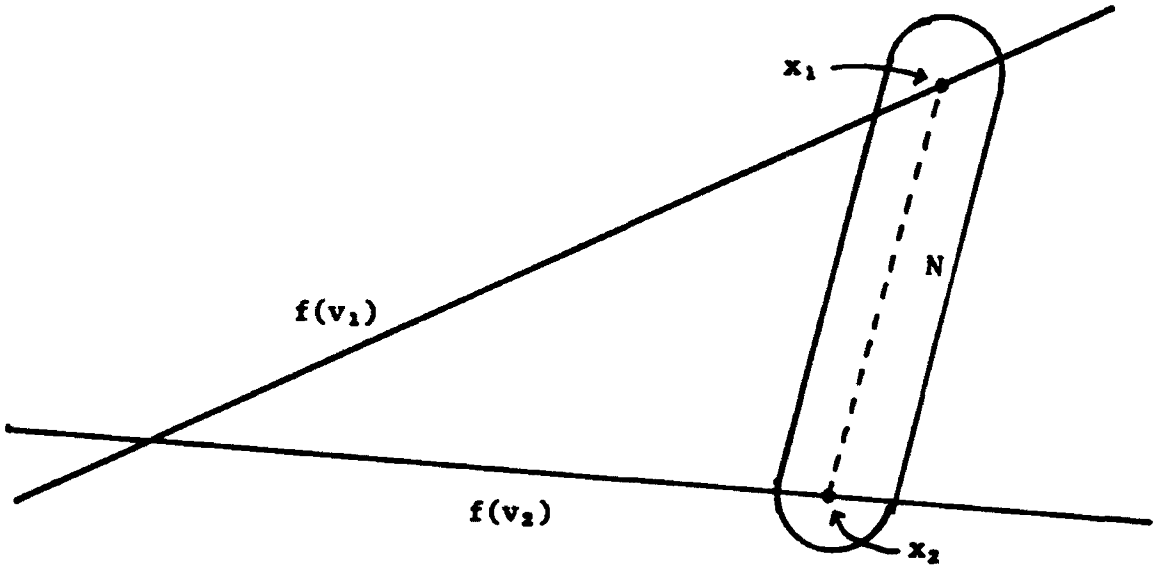


Figure 5.17

(2) $N \cap f(w)=\emptyset$ for $w \neq v_i$, i=1,2. See figure 5.17.

5.6.3 Definition. Let f be a planar multiple line segment representation for a plane graph G. We say that f is a **P-special planar multiple line segment representation** if:

(1) if v is an outervertex of G, then f(v) is a (single) line segment,

(2) if v is an innervertex of G, then f(v) is the union of (at most) two line segments,

(3) all vertices are displayed in f, and

(4) all edges of oG are displayed in f.

5.6.4 Definition. Let $L_1, L_2, \ldots, L_t$ be planar line segments. We say these line segments form a **comb arrangement** provided they are arranged as in figure 5.18. The defining property of this arrangement is the

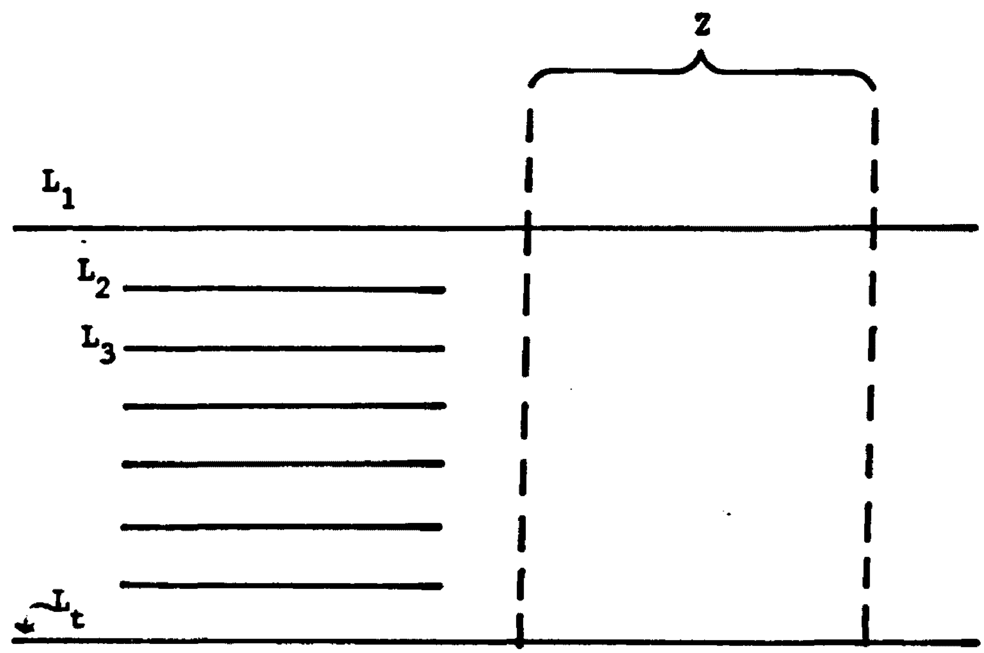


Figure 5.18

interaction zone (denoted Z in the figure) which has the property that the right endpoints of $L_2, \ldots, L_{t-1}$ can be extended right into the interaction zone without forming new intersections. Although the line segments in the figure are drawn parallel and horizontal, neither of these properties is necessary for our definition.

5.6.5 Theorem. If G is a plane graph, then G has a P-special planar multiple line segment representation, hence $G \in \Omega(2\mathrm{LS}^2)$.

Proof. Let G be a plane graph. If G is a single vertex the result is obvious and so we assume the theorem is true for all plane graphs with fewer than $|V(G)|$ vertices. As usual, we may assume G is connected.

Suppose oG contains a vertex v of degree 1, and let w be its neighbor. Let f be a P-special planar multiple line segment representation for G-v. We extend f to be a representation for G by assigning to v a short line segment which intersects w and is contained in w's display disc as in figure 5.19. One checks that this gives a P-special planar multiple line segment representation for G, so we may now

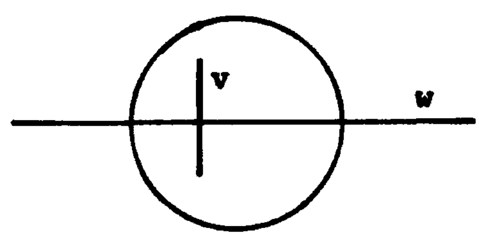


Figure 5.19

assume oG has no vertices of degree 1.

As in the proof of 5.3.9 and 5.5.5 we can find a leaf face C of oG with vertices $v_0, v_1, \ldots, v_k$ (where $k \geq 2$) so that $d_{oG}(v_i)=2$ for $2 \leq i \leq k$. Let f be a P-special planar multiple line segment representation for $G'=G-\{v_2, \ldots, v_k\}$. We use the ε-neighborhood N which spans from $f(v_0)$ to $f(v_1)$ to reconstruct the cycle $v_0, v_1, v_2, \ldots, v_k$ as shown in figure 5.20.

Observe that $f(v_1), f(v_2), \ldots, f(v_k)$ form a comb arrangement with interaction zone Z, as shown. Notice that all vertices v_i and all edges $v_i v_{i+1}$ are displayed. Vertices internal and adjacent to cycle C are outervertices in G' and we may assign a second line segment to each in extending f to G.

Let u be contained within cycle C and adjacent to either one v_i with $2 \leq i \leq k$ or to two such adjacent vertices. Note that we need not be concerned if u is adjacent to either v_0 or v_1 since these vertices are in G'. One assigns to u a second line segment as shown in figure 5.21.

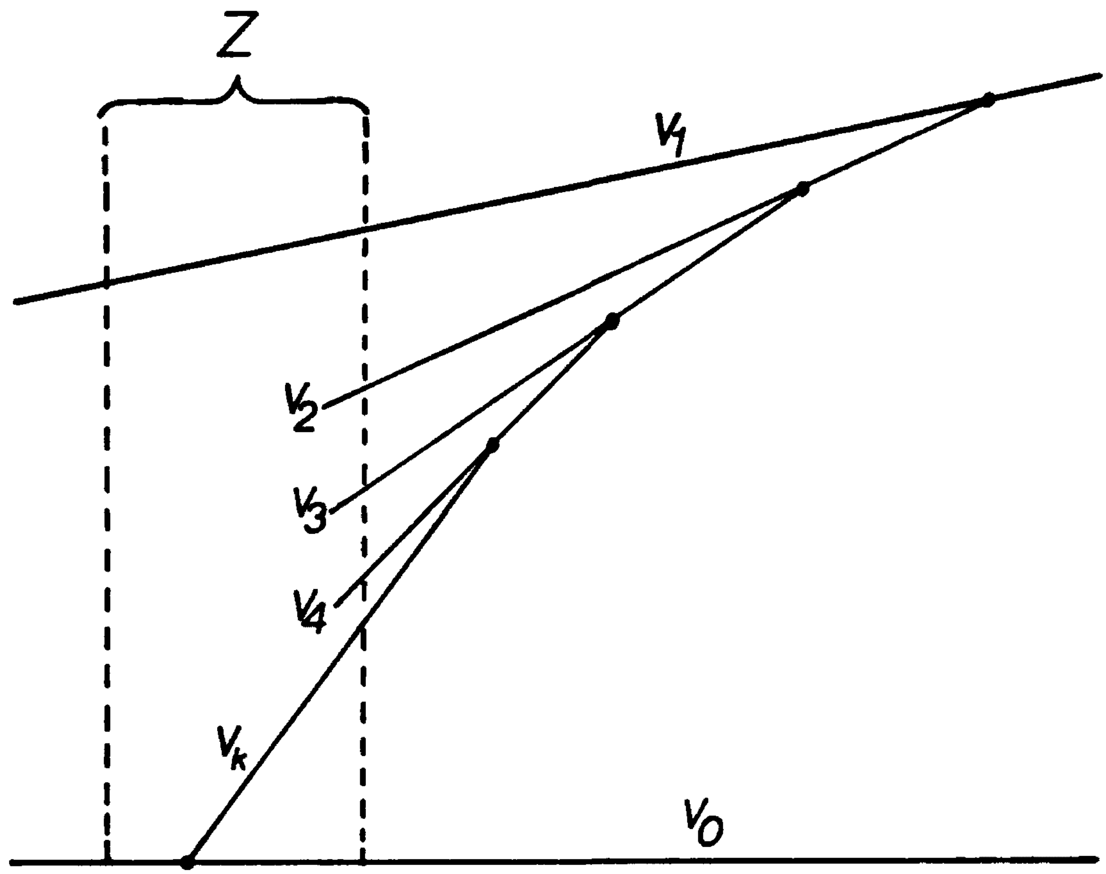


Figure 5.20

We may therefore assume all remaining innervertices adjacent to vertices on C meet v_i and v_j with $2 \le i < j-1$. Let u be such a vertex and let $C_u = \{v_i : u \sim v_i \text{ and } i \ge 2\} = \{v_{i_1}, v_{i_2}, \ldots, v_{i_p}\}$ where $i_j < i_{j+1}$. Extend the line segments for the vertices in C_u partially into the interaction zone and place a second segment for u as shown in figure 5.22. The second segment for u may extend to meet the segment for v_k in case $u \sim v_k$. Observe that

$f(v_1), f(v_2), \ldots, f(v_{i_1}), f(v_{i_p}), \ldots, f(v_k)$ and

$f(v_{i_1}), f(v_{i_1+1}), \ldots, f(v_{i_2})$ and

$f(v_{i_2}), f(v_{i_2+1}), \ldots, f(v_{i_3})$ and ...

and ... and

$f(v_{i_{p-1}}), f(v_{i_{p-1}+1}), \ldots, f(v_{i_p})$

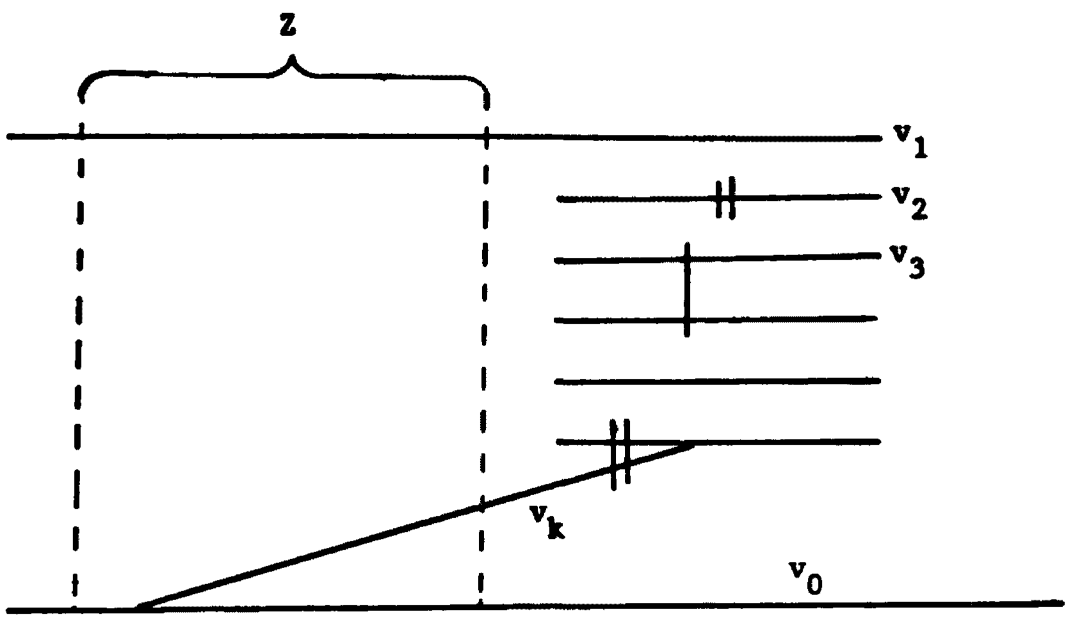


Figure 5.21

all form comb arrangements. The pie-slice lemma assures us that any second vertex u' which meets vertices in C can have a second line segment appended to the representation using one of the aforementioned combs. Recursively continuing in this fashion (see figure 5.23) we form a P-special planar multiple line segment representation for G.∎

5.6.6 Corollary. If G is an outerplanar graph, then $G \in \Omega(\mathbf{LS}^2)$.∎

5.6.7 Remark. It is unknown if Theorem 5.6.5 is best possible. It is not known if there exists a planar graph which requires two planar line segments per vertex in order to be represented, or if all planar graphs are in $\Omega(\mathbf{LS}^2)$.

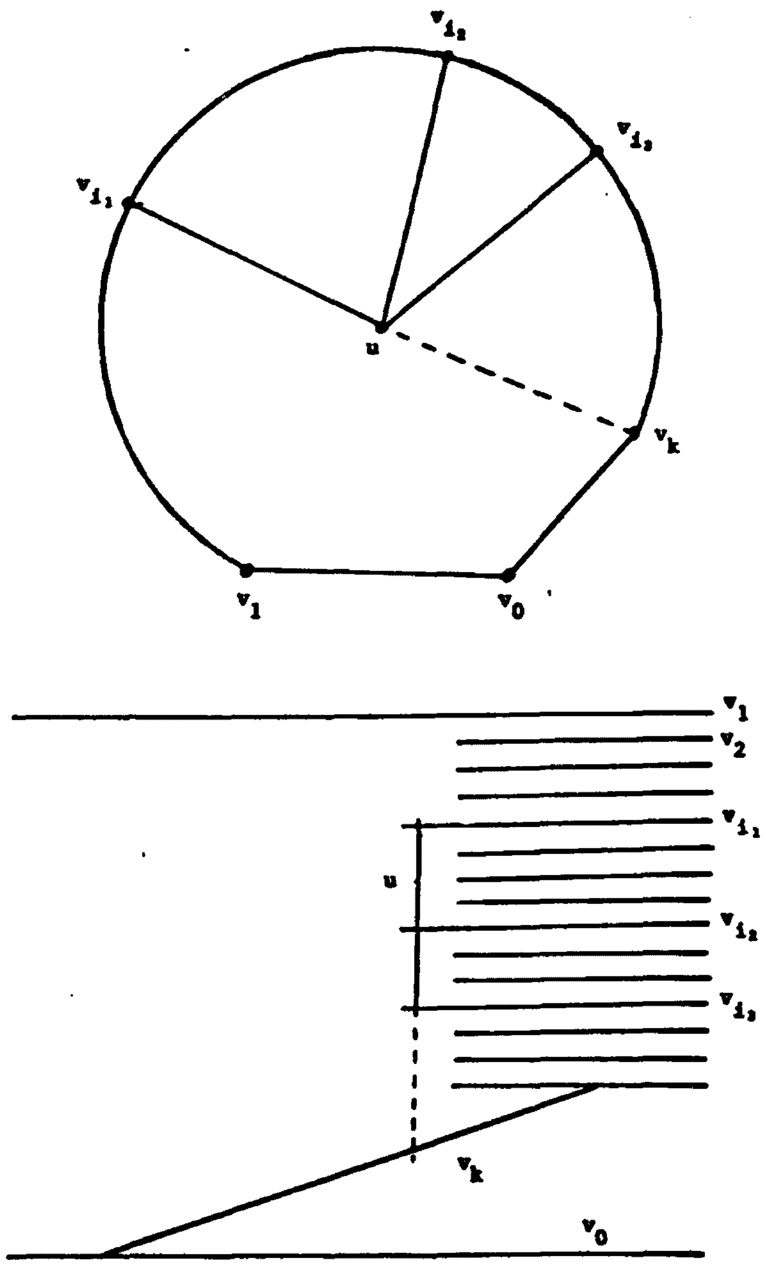


Figure 5.22

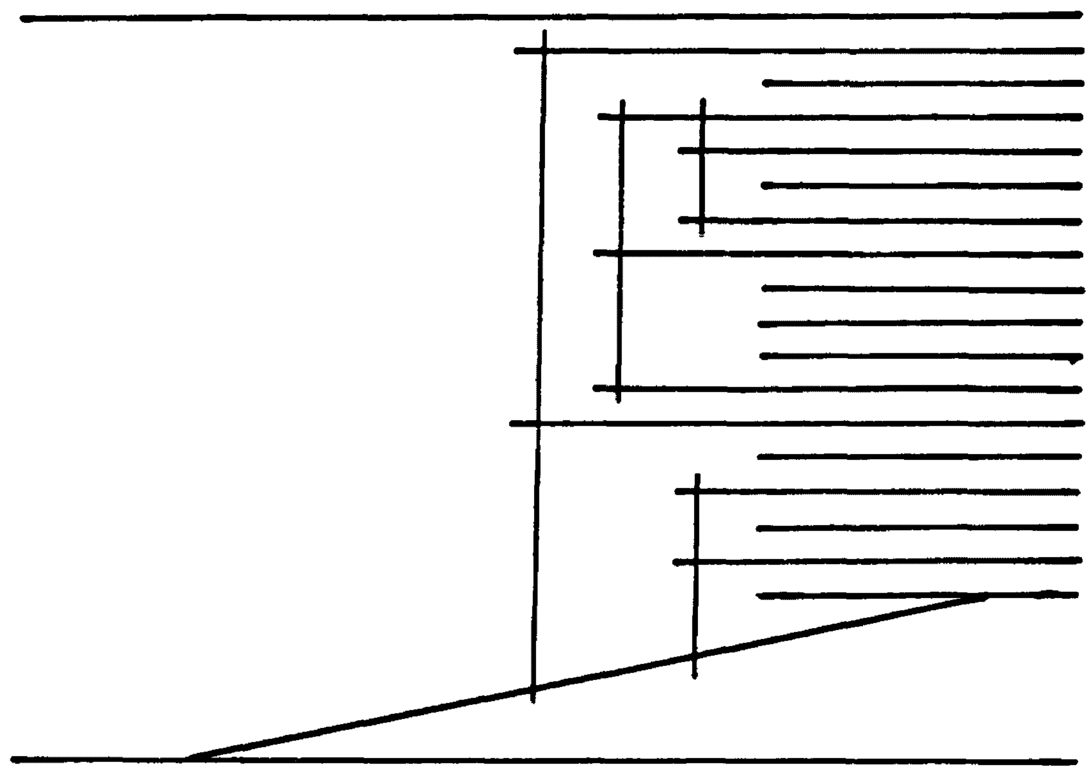


Figure 5.23

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